Complex Manifolds

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These notes are produced entirely from the course I took, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. Please send any corrections to pdtwm2@cam.ac.uk

Recommended books: Huybrechts, *Complex Geometry: An Introduction*.

Contents

0. Introduction and Motivation

Complex geometry is the study of complex manifolds, which are the holomorphic version of smooth manifolds. These locally look like open subsets of $\mathbb{C}^n$, with holomorphic transition functions.

One dimensional complex manifolds are Riemann surfaces. Every (smooth) projective variety is a complex manifold. A main result of this course gives a partial converse to this (and on the first example sheet we shall see an example of a complex manifold which is not algebraic).

Complex tools are often used to study projective varieties (Hodge conjecture, Moduli theory). There are also lots of questions which are also interesting in their own right. Projective surfaces were classified in 1916. The classification of compact complex surfaces is still open (most recent progress was in 2005).

0.1. **Several Complex Variables.**

Definition 0.1. *Let $U \subset \mathbb{C}^n$ be open. Then a smooth function $f : U \to \mathbb{C}$ is* ***holomorphic*** *if it is holomorphic in each variable (i.e. fix all z_i but one, then consider f as a function of that one $z_j \in \mathbb{C}$).*

A function $F : U \to \mathbb{C}^m$ is ***holomorphic*** *if each coordinate function is holomorphic.*

Remark: There are equivalent definitions of holomorphicity in terms of existence of power series.

Now identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ via $(x_1+iy_1,\dots,x_n+iy_n) \mapsto (x_1,y_1,\dots,x_n,y_n)$. Then if we write $f = u+iv$ in terms of its real and imaginary parts, basic complex analysis implies:

$$f \text{ is holomorphic} \iff \frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \quad \text{and} \quad \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j} \quad \forall j$$

(i..e the Cauchy-Riemann conditions hold). More conveniently, if we define

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

then

$$f \text{ is holomorphic} \iff \frac{\partial f}{\partial \overline{z}_j} = 0 \quad \forall j.$$

Proposition 0.1 (Maximum Principle). *Let $U \subset \mathbb{C}^n$ be open and connected. Suppose f is holomorphic on U, and that D is open and bounded with $\overline{D} \subset U$. Then:*

$$\max_{\overline{D}} |f| = \max_{\partial \overline{D}} |f|.$$

Proof. Repeated application of the single variable maximum principle from complex analysis. □

So in particular the maximum principle tells us that if $|f|$ attains its maximum at an interior point, then f must be constant.

Proposition 0.2 (Identity Principle)**.** *Suppose $U \subset \mathbb{C}^n$ is open and connected, with $f : U \to \mathbb{C}$ holomorphic. Suppose f vanishes on an open subset of U. Then $f \equiv 0$.*

Proof. Repeated application of the single variable identity principle from complex analysis. $\square$

1. Complex Manifolds

Let X be a second countable, Hausdorff, topological space. We always assume that X is connected (e.g. a smooth manifold).

Definition 1.1. *A* ***holomorphic atlas*** *for X is a collection of charts $(U_\alpha, \varphi_\alpha)$ with $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$ homeomorphisms such that:*

(i) $X = \cup_\alpha U_\alpha$ is an open cover

(ii) The transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic.

Definition 1.2. *Two holomorphic atlases $(U_\alpha, \varphi_\alpha)_\alpha$, $(\tilde{U}_\beta, \tilde{\varphi}_\beta)$ are* ***equivalent*** *if $\varphi_\alpha \circ \tilde{\varphi}_\beta^{-1}$ are holomorphic for all α, β.*

i.e. if their union is also an atlas.

Definition 1.3. *A* ***complex manifold*** *is a topological space as above with an equivalence class of holomorphic atlases (i.e. a maximal atlas).*

Such an equivalence class is called a ***complex structure****.*

Example 1.1. *$\mathbb{C}^n$ is a complex manifold. Moreover any open subset of $\mathbb{C}^n$ is a complex manifold, e.g. the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.*

Example 1.2 (Complex Projective Space)**.** *Consider (complex) projective space $\mathbb{P}^n$. As a set this is the linear 1-dimensional subspaces of $\mathbb{C}^{n+1}$. A point in $\mathbb{P}^n$ is represented by $[z_0 : \cdots : z_n]$.*

A holomorphic atlas is given by $U_i = \{z_i \neq 0\}$ with φ_i defined by:

$$\varphi_i([z_0 : \cdots : z_n]) := \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right)$$

where as usual a 'hat' means we omit that term. One can then check that the transition functions are holomorphic and so $\mathbb{P}^n$ is a complex manifold. Moreover we can see that $\mathbb{P}^n$ is a <u>compact</u> *complex manifold.*

Definition 1.4. *A smooth function $f : X \to \mathbb{C}$ (X a complex manifold) is said to be* ***holomorphic*** *if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic for all charts (U, φ).*

Definition 1.5. *A smooth map $F : X \to Y$ between complex manifolds X, Y is* ***holomorphic*** *if for all charts (U, φ) for X and (V, ψ) for Y, the map $\psi \circ F \circ \varphi^{-1}$ is holomorphic.*

We see that F is ***biholomorphic*** *if it has a holomorphic inverse.*

Exercise: (Extension of maximum principle). If X is compact, show that any holomorphic function on X is constant.

Thus compact complex manifolds cannot be embedded into $\mathbb{C}^m$ for any m. Contrast this with the smooth manifold case, where Whitney's embedding theorem tells us that we can always embed smooth manifolds into some $\mathbb{R}^m$. So complex manifold theory is very different.

Exercise: (Extension of Identity Principle). If $X \to \mathbb{C}$ is holomorphic and vanishes on an open set in X, then $f \equiv 0$.

Thus there are no holomorphic analogues of bump functions or partitions of unity in complex manifolds, again making them very different to smooth manifold theory.

Definition 1.6. *Let $Y \subset X$ be a smooth submanifold of dimension $2k < 2n = \dim(X)$. Then we say that Y is a* ***closed complex submanifold*** *if $\exists$ a holomorphic atlas for X such that $\varphi_\alpha : U_\alpha \cap Y \to \varphi(U_\alpha) \cap \mathbb{C}^k$, where $\mathbb{C}^k \subset \mathbb{C}^n$ is identified by $(z_1, \ldots, z_k, 0, \ldots, 0)$.*

Exercise: Show that a closed complex submanifold is naturally a complex manifold.

Definition 1.7. *We say a complex manifold X is* ***projective*** *if it is biholomorphic to a compact closed complex submanifold of $\mathbb{P}^m$ for some m.*

Theorem 1.1 (Chow)**.** *A projective complex manifold is actually a projective variety.*

Proof. Later. □

Recall that a variety is the vanishing set of some polynomial equations over some space. So a projective variety is the vanishing set in $\mathbb{P}^m$ of some homogeneous polynomial equations (as in $\mathbb{P}^m$).

In the example sheet we will see an example of a compact complex manifold which is not projective.

1.1. **Almost Complex Structures.**

Before we work globally on manifolds we need to understand how to work on them locally, and thus we need to consider the linear space case first.

So let V be a real vector space.

Definition 1.8. *A linear map $J : V \to V$ with $J^2 = -\mathrm{id}$ is called a* ***complex structure****.*

On $\mathbb{R}^{2n}$, the endomorphism $(x_1, y_1, \ldots, x_n, y_n) \xrightarrow{J} (-y_1, x_1, \ldots, -y_n, x_n)$ is called the **standard complex structure** (this just comes from multiplication by i, as $x_j + iy_j \underset{\times i}{\to} -y_j + ix_j$).

Now as $J^2 = -\mathrm{id}$ for any complex structure, the eigenvalues are $\pm i$, and so since V is real there are no (real) eigenspaces. To get around this, we consider the **complexification of V**, defined by

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}.$$

Then J extends to $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ with $J^2 = -\mathrm{id}$ via:

$$J(v \otimes z) := J(v) \otimes z.$$

So let $V^{1,0}$ and $V^{0,1}$ denote the eigenspaces in $V_{\mathbb{C}}$ of $\pm i$ respectively.

Lemma 1.1. *For V a real vector space and J a complex structure on V, we have:*

(i) $V_{\mathbb{C}} = V^{1,0} \otimes V^{0,1}$.

(ii) $\overline{V^{1,0}} = V^{0,1}$, *where $\overline{(\cdot)}$ denotes the conjugate.*

Proof. (i): For $v \in V_{\mathbb{C}}$ we can write:

$$v = \underbrace{\frac{1}{2}(v - iJ(v))}_{\in V^{1,0}} + \underbrace{\frac{1}{2}(v + iJ(v))}_{\in V^{0,1}}$$

and so $V_{\mathbb{C}} = V^{1,0} + V^{0,1}$. But then clearly the eigenspaces are disjoint (except for 0) and so $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.

(ii): Follows from the decomposition in (i), as taking complex conjugates of $V^{1,0}$ part gives something in the $V^{0,1}$ part.

$\square$

Now to look at the case of manifolds:

Definition 1.9. *Let X be a smooth manifold. Then an* ***almost complex structure (a.c.s)*** *on X is a bundle isomorphism $J : TX \to TX$ with $J^2 = -\mathrm{id}$ (i.e. $J_x : T_xX \to T_xX$ for all $x \in X$ with $J_x^2 = -\mathrm{id}_x$).*

One can complexify TX to obtain $(TX)_{\mathbb{C}} = TX \otimes \mathbb{C}$ (this is really a tensor product of two vector bundles, where by $\mathbb{C}$ we mean the trivial bundle with fibre $\mathbb{C}$ at each point, i.e. $(TX \otimes C)_p = T_pX \otimes \mathbb{C}$ for all p).

So each fibre of the bundle $(TX)_{\mathbb{C}} \to X$ is a complex vector space. We call $(TX)_{\mathbb{C}}$ the **complexified tangent bundle**.

Then as we saw above, we can split each fibre up into its eigenvalue decomposition, and so we see that $(TX)_{\mathbb{C}}$ splits as a direct sum:

$$(TX)_{\mathbb{C}} \cong (TX)^{(1,0)} \oplus (TX)^{(0,1)}.$$

Fibrewise this is exactly as above. To obtain this as vector bundles though, one can use, e.g.

$$(TX)^{(1,0)} = \ker(J - i \cdot \mathrm{id}) \quad \text{and} \quad (TX)^{(0,1)} = \ker(J + i \cdot \mathrm{id}).$$

To see that complex manifolds naturally have an a.c.s, we need the following:

Exercise: [See Example Sheet 1]. Let $U, V \subset \mathbb{C}^n$ be open and $f : U \to V$ be smooth. Then:

$$f \text{ is holomorphic} \quad \Longleftrightarrow \quad \mathrm{d}f \text{ is } \mathbb{C}\text{-linear.}$$

Now on $T\mathbb{R}^{2n}$ there is a natural a.c.s, denoted J_{st} ("st" for "standard")m coming from the complex structure on $\mathbb{R}^{2n}$ we saw before.

So let X be a complex manifold. Then if $U \subset X$ is a chart, $\varphi : U \to \varphi(U) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ is a biholomorphism and so the differential of φ gives a bundle map $J : TU \to TU$ defined by: $J := \mathrm{d}\varphi^{-1} \circ J_{\mathrm{st}} \circ \mathrm{d}\varphi$.

So we have defined a local a.c.s on a complex manifold (just by simply pulling the one on $\mathbb{R}^{2n}$ back). To see that we can patch these local a.c.s's together to give an a.c.s on all of X, we just need to check that the above local definition is independent of the choice of chart.

Proposition 1.1. *The a.c.s J defined above is independent of the choice of (holomorphic) chart, and thus gives an a.c.s on X.*

Proof. Suppose φ, ψ are charts around the same point. What we need to show is that:

$$\mathrm{d}\varphi^{-1} \circ J_{\mathrm{st}} \circ \mathrm{d}\varphi = \mathrm{d}\psi^{-1} \circ J_{\mathrm{st}} \circ \mathrm{d}\psi \qquad \text{i.e.} \qquad \mathrm{d}\big((\varphi \circ \psi^{-1})^{-1}\big) \circ J_{\mathrm{st}} \circ \mathrm{d}\big(\varphi \circ \psi^{-1}\big) = J_{\mathrm{st}}.$$

Now $\varphi \circ \psi^{-1}$ is a holomorphic map between open subsets of $\mathbb{C}^n$, and so $\mathrm{d}\big((\varphi \circ \psi^{-1})\big)$ commutes with J_{st}[(i)]. So thus the above equality does hold, and so we are done.

□

Remark: There are lots of a.c.s's that do not arise from a complex manifold structure. The a.c.s's that do arise from a complex structure are called **integrable**. For example, S^6 admits an a.c.s which is not integrable, i.e. is not induced by a complex structure on S^6. It is still an open problem to determine whether or not S^6 admits a complex structure or not.

A general result in complex manifold theory gives a condition for when an a.c.s is integrable:

$$\text{An a.c.s is integrable} \quad \Longleftrightarrow \quad \text{The Nijenhuis tensor vanishes.}$$

[(i)] This is seen from the proof of the above exercise on f being holomorphic iff $\mathrm{d}f$ is $\mathbb{C}$-linear. It also shows (or at least a very similar argument) that f is holomorphic iff f commutes with J_{st}, i.e. $\mathrm{d}f \circ J = J \circ \mathrm{d}f$.

Definition 1.10. *$TX^{(1,0)}$ is called the* ***holomorphic tangent bundle of X****.*

Now if V is a real vector space, and if J is a complex structure on V, then one obtains a complex structure on V^* via:

$$(J^*\alpha)(v) := \alpha(J(v)) \qquad \forall \alpha \in V^*.$$

Then analogously to what we have seen above one obtains a decomposition of the **complexified cotangent bundle**:

$$(T^*X)_{\mathbb{C}} \cong T^*X^{(1,0)} \oplus T^*X^{(0,1)}$$

where $(T^*X)_{\mathbb{C}} := T^*X \otimes \mathbb{C}$. So locally if $\varphi : U \to \mathbb{C}^n$ is a chart then we say that $z_j = x_j + iy_j$ are local coordinates in U (where $\varphi = (z_1, \ldots, z_n)$).

In these local coordinates, we can then see (for the complexified tangent bundle, as we have a local basis $\left\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\}_j$):

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \qquad \text{and} \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

(from look back at the form of the standard complex structure on $\mathbb{R}^{2n}$ to see this), and then for the complexified cotangent bundle this gives (just using the above formula):

$$J^*(\mathrm{d}x_j) = -\mathrm{d}y_j \qquad \text{and} \qquad J(\mathrm{d}y_j) = \mathrm{d}x_j.$$

[These just come from the usual differential geometry calculations, e.g.

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}(J) = \frac{\partial}{\partial x_j}(-y_1, x_1, \ldots, -y_n, x_n) = (0, \ldots, 0, \underbrace{1}_{y_j \text{ place}}, 0, \ldots, 0) = \frac{\partial}{\partial y_j}$$

and simiarly for $J\left(\frac{\partial}{\partial y_j}\right)$. Then the dual expressions come from the definition of J^* and the fact that $\{\mathrm{d}x_j, \mathrm{d}y_j\}_j$ is the dual basis.]

So we know what the a.c.s looks like in local coordinates.

Definition 1.11. *We define:*

$$\mathrm{d}z_j := \mathrm{d}x_j + i\mathrm{d}y_j \quad \textit{and} \quad \mathrm{d}\overline{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j$$

as well as

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j}\right) \quad \textit{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Then $\mathrm{d}z_j, \mathrm{d}\overline{z}_j$ are sections of $(T^*X)_{\mathbb{C}}$ and $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}$ are sections of $(TX)_{\mathbb{C}}$, which we dual to one another in the usual sense of differential geometry.

Note: We can readily check that:

$$J(\mathrm{d}z_j) = i\mathrm{d}z_j, \quad J(\mathrm{d}\overline{z}_j) = -i\mathrm{d}\overline{z}_j, \quad J\left(\frac{\partial}{\partial z_j}\right) = i\frac{\partial}{\partial z_j}, \quad J\left(\frac{\partial}{\partial \overline{z}_j}\right) = -i\frac{\partial}{\partial \overline{z}_j}.$$

We see from this that the $\mathrm{d}z_j$ form a local frame/basis for $T^*X^{(1,0)}$ and similarly the $\mathrm{d}\overline{z}_j$ form a local frame for $T^*X^{(0,1)}$. Also exactly the same holds for $TX^{(1,0)}, TX^{(0,1)}$ with the $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}$.

Now if $f : X \to \mathbb{C}$ is a smooth function with $f = u + iv$, then $\mathrm{d}f = \mathrm{d}u + i\mathrm{d}v$ is a smooth section of $(T^*X)_{\mathbb{C}} \cong T^*X^{(1,0)} \oplus T^*X^{(0,1)}$. Now if we write p_1, p_2 for the two projections from $(T^*X)_{\mathbb{C}}$ onto $T^*X^{(1,0)}$ and $T^*X^{(0,1)}$ respectively, then we define the **del** and **del-bar operators** ($\boldsymbol{\partial}$, $\boldsymbol{\overline{\partial}}$) by

$$\partial f := p_1(\mathrm{d}f) \qquad \text{and} \qquad \overline{\partial} := p_2(\mathrm{d}f).$$

In a local frame just as we expect we have

$$\begin{aligned} \mathrm{d}f &= \underbrace{\sum_j \frac{\partial f}{\partial z_j} \cdot \mathrm{d}z_j}_{=\partial f} + \underbrace{\sum_j \frac{\partial f}{\partial \overline{z}_j} \cdot \mathrm{d}\overline{z}_j}_{=\overline{\partial} f} \\ &= \partial f + \overline{\partial} f \end{aligned}$$

and so on smooth functions, $\mathrm{d} = \partial + \overline{\partial}$. Thus we see

$$f \text{ is holomorphic} \quad \Longleftrightarrow \quad \overline{\partial} f = 0.$$

This is all we need for 1-forms, so now we can do the same for higher degree forms. Write

$$\Lambda^{p,q}(T^*X) := \Lambda^p\left(T^*X^{(1,0)}\right) \otimes \Lambda^q\left(T^*X^{(0,1)}\right)$$

where Λ^p denotes the p'th exterior power.

Definition 1.12. *A section of $\Lambda^{p,q}(T^*X)$ is called a **(p,q)-form**.*

Locally a (p,q)-form looks like:

$$\sum_{J,L} f_{JL} \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}z_{j_p} \wedge \mathrm{d}\overline{z}_{l_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{l_q}$$

where $J = (j_1, \ldots, j_p)$, $L = (l_1, \ldots, l_q)$ and the sum is over all such multi-indices. Here the f_{JL} are just smooth functions. Thus, e.g. $\overline{z}\mathrm{d}z$ is a section of $(T^*X)^{(1,0)}$, despite the coefficient not being holomorphic. Thus we do not require the coefficients in a (p,q)-form to be holomorphic or anti-holomorphic, they are just smooth functions.

Definition 1.13. *We write $\boldsymbol{\mathscr{A}^k_{\mathbb{C}}(U)}$ for the smooth sections of $\Lambda^k\left((T^*X)_{\mathbb{C}}\right)$ over $U \subset X$, i.e. complexified k-forms.*

We also write $\boldsymbol{\mathscr{A}^{p,q}_{\mathbb{C}}(U)}$ for the smooth sections of $\Lambda^{p,q}(U)$.

So$\mathscr{A}^{0,0}_{\mathbb{C}}(U)$ consists of the smooth $\mathbb{C}$-valued functions on U. We often omit the subscript $\mathbb{C}$ as it is understood.

Lemma 1.2 (Relation between (p,q)-forms and k-forms)**.**

(i) There is a natural identification

$$\Lambda^k\left((T^*X)_{\mathbb{C}}\right) \cong \bigoplus_{p,q:\ p+q=k} \Lambda^{p,q}(T^*X).$$

So in particular the same is true for the space of sections:

$$\mathscr{A}^k_{\mathbb{C}}(U) \cong \bigoplus_{p+q=k} \mathscr{A}^{p,q}_{\mathbb{C}}(U).$$

(ii) If $\alpha \in \mathscr{A}^{p,q}_{\mathbb{C}}(U)$, $\beta \in \mathscr{A}^{p',q'}_{\mathbb{C}}(U)$, *then* $\alpha \wedge \beta \in \mathscr{A}^{p+p',q+q'}_{\mathbb{C}}(U)$.

Proof. Fibrewise this is just linear algebra. One can then use the local frame/coordinates to obtain the result on bundles.

□

So now we are in a good position to define a cohomology theory on complex manifolds.

1.2. **Dolbeault Cohomology.**

Denote by $\mathrm{d} : \mathscr{A}^k_{\mathbb{C}}(U) \to \mathscr{A}^{k+1}_{\mathbb{C}}(U)$ the usual exterior derivative.

Definition 1.14. *Define the* ***del operator*** *on* (p,q)*-forms* $\partial : \mathscr{A}^{p,q}_{\mathbb{C}}(U) \to \mathscr{A}^{p+1,q}_{\mathbb{C}}(U)$ *by:*

$$\partial = \pi_1 \circ d$$

i.e. taking d *composed with the projection onto* $(p+1,q)$*-closed forms* $\pi_1 : \mathscr{A}^{p+q+1}_{\mathbb{C}}(U) \to \mathscr{A}^{p+1,q}_{\mathbb{C}}(U)$.

Similarly define the ***del-bar operator*** *on* (p,q)*-forms* $\overline{\partial} : \mathscr{A}^{p,q}_{\mathbb{C}}(U) \to \mathscr{A}^{p,q+1}_{\mathbb{C}}(U)$ *by:*

$$\overline{\partial} = \pi_2 \circ d$$

for π_2 *the projection* $\pi_2 : \mathscr{A}^{p+q+1}_{\mathbb{C}}(U) \to \mathscr{A}^{p,q+1}_{\mathbb{C}}(U)$.

Note: These projection operators are well-defined by Lemma 1.2.

Definition 1.15. *The* (p,q)*-**Dolbeault cohomology** of* X *is given by:*

$$H^{p,q}_{\overline{\partial}}(X) := \frac{\ker\left(\overline{\partial} : \mathscr{A}^{p,q}_{\mathbb{C}}(X) \to \mathscr{A}^{p,q+1}_{\mathbb{C}}(X)\right)}{\mathrm{Image}\left(\overline{\partial} : \mathscr{A}^{p,q-1}_{\mathbb{C}}(X) \to \mathscr{A}^{p,q}_{\mathbb{C}}(X)\right)}.$$

To check that this is well-defined we need to see that $\overline{\partial}^2 = 0$. Note first that, locally, if

$$\alpha = \sum_{J,L} f_{JL} \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}z_{j_p} \wedge \mathrm{d}\overline{z}_{l_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{l_q}$$

then

$$\mathrm{d}\alpha = \underbrace{\sum_{J,L}\sum_{r} \frac{\partial f}{\partial z_r} \mathrm{d}z_r \wedge \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{l_q}}_{=\partial\alpha} + \underbrace{\sum_{J,L}\sum_{r} \frac{\partial f}{\partial \overline{z}_r} \mathrm{d}\overline{z}_r \wedge \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{l_q}}_{\overline{\partial}\alpha}$$

where we have split the sum up into two parts, depending on whether we get an extra $\mathrm{d}z_r$ or $\mathrm{d}\overline{z}_r$.

From this we can establish:

Lemma 1.3 (Properties of $\partial, \overline{\partial}$ for complex manifolds).

(i) $\mathrm{d} = \partial + \overline{\partial}$.

(ii) $\partial^2 = 0 = \overline{\partial}^2$ *and* $\partial\overline{\partial} = -\overline{\partial}\partial$.

(iii) If $\alpha \in \mathscr{A}_{\mathbb{C}}^{p,q}(U)$, $\beta \in \mathscr{A}_{\mathbb{C}}^{p',q'}(U)$, *then:*

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \mathrm{d}\beta$$
$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial}\beta.$$

Proof. (i): Follows from the local coordinate expressions as defined and used above.

(ii): Since $\mathrm{d} = \partial + \overline{\partial}$ and $\mathrm{d}^2 = 0$ we have

$$0 = \mathrm{d}^2 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2.$$

But note that ∂^2 maps into $\mathscr{A}^{p+2,q}$, $\partial\overline{\partial}$ and $\overline{\partial}\partial$ map into $\mathscr{A}^{p+1,q+1}$ and $\overline{\partial}^2$ maps into $\mathscr{A}^{p,q+2}$. Since all of these spaces are disjoint (except for 0), the above equality can only be true if we have $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$ separately, which gives the result.

(iii): This simply follows from $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \mathrm{d}\beta$, which is because the total rank of α is $p+q$.

□

Thus since $\overline{\partial}^2 = 0$ this tells us that $\left(\mathscr{A}_{\mathbb{C}}^{p,\star}(X), \overline{\partial}\right)$ is a cochain complex for each p, and so the (p,q)-Dolbeault cohomology groups are well-defined. Note that they are also vector spaces.

Remark: One could make an analogous definition using ∂ instead of $\overline{\partial}$. However the information would be equivalent just by complex conjugating. We tend to work with $\overline{\partial}$ simply because we like holomorphic things, and for smooth functions f we know: $f \in \ker(\overline{\partial}) \iff f$ is holomorphic.

Recall: In differential geometry we define the **de Rham cohomology**

$$H^i_{\mathrm{dR}}(X;\mathbb{R}) := \frac{\ker\left(\mathrm{d} : \mathscr{A}^i_{\mathbb{R}}(X) \to \mathscr{A}^{i+1}_{\mathbb{R}}(X)\right)}{\mathrm{Image}\left(\mathrm{d} : \mathscr{A}^{i-1}_{\mathbb{R}}(X) \to \mathscr{A}^i_{\mathbb{R}}(X)\right)}.$$

One similarly defines the de Rham cohomology for the complexification:

$$\begin{aligned} H^i_{\mathrm{dR}}(X;\mathbb{C}) &:= \frac{\ker\left(\mathrm{d} : \mathscr{A}^i_{\mathbb{C}}(X) \to \mathscr{A}^{i+1}_{\mathbb{C}}(X)\right)}{\mathrm{Image}\left(\mathrm{d} : \mathscr{A}^{i-1}_{\mathbb{C}}(X) \to \mathscr{A}^i_{\mathbb{C}}(X)\right)} \\ &\cong H^i_{\mathrm{dR}}(X;\mathbb{R}) \otimes \mathbb{C}. \end{aligned}$$

Much of this course will prove the Hodge decomposition for a certain class of compact complex manifolds, which includes projective varieties. It says that:

$$H^i_{\mathrm{dR}}(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(X)$$

(note that this is for the complexified de Rham cohomology). This result is not true for general complex manifolds (e.g. the Hopf surface).

Exercise: If $f : X \to Y$ is holomorphic, show that f induces a map

$$f^* : H^{p,q}_{\overline{\partial}}(Y) \to H^{p,q}_{\overline{\partial}}(X)$$

by pullback.

Example 1.3 (Motivation for why we might care about Dolbeault cohomology - The Mittag-Leffler Problem)**.**

Let S be a Riemann surface (i.e. a one-dimensional complex manifold). Then a ***principal part*** *at $x \in S$ is a Laurent series of the form $\sum_{k=1}^n a_k z^{-k}$, with z a local coordinate about x in S. The Mittag-Leffler problem asks:*

"Given points $x_1, \ldots, x_n \in S$ and principal parts $P_1, \ldots, P_n$, is there a meromorphic function on S with principal part P_i at x_i for all i?"

(By meromorphic function on S we mean a holomorphic map $S \to \mathbb{P}^1$ of complex manifolds, or just a locally meromorphic map on S in the usual sense of complex analysis.)

To do this, take local solutions f_i at x_i defined on U_i (defined by the principal parts) and take a smooth partition of unity $(\rho_i)_i$ subordinate to the $(U_i)_i$. Then we know $\sum_{j=1}^n \rho_j f_j$ is <u>smooth</u> on $S\backslash\{x_1, \ldots, x_n\}$, with the desired local expression at each x_i (since $\rho_i \equiv 1$ about x_i whilst all others are 0). We need to know if this is holomorphic on $S\backslash\{x_1, \ldots, x_n\}$ though.

A calculation then shows that $g = \overline{\partial}\left(\sum_j \rho_j f_j\right)$ extends to a smooth $(0,1)$-form on S. Clearly $\overline{\partial} g = 0$ as $\overline{\partial}^2 = 0$, and so $[g] \in H^{0,1}_{\overline{\partial}}(S)$. So <u>suppose</u> $H^{0,1}_{\overline{\partial}}(S) = 0$. Then this implies $\exists$ a smooth function h with $\partial h = g$, and so if we define $f := \sum_j \rho_j f_j - h$, then $\overline{\partial} f = 0$, and so f solves the Mittag-Leffler problem.

It turns out that this condition is an iff, i.e.

$$\text{We can solve the Mittag-Leffler problem on } S \iff H^{(0,1)}_{\overline{\partial}}(S) = 0.$$

The converse implication requires knowledge of sheaf cohomology, which we will study shortly.

1.3. **The $\overline{\partial}$-Poincaré Lemma.**

Recall that if X is a contractible smooth manifold, then $H^i_{\mathrm{dR}}(X;\mathbb{R}) = 0$ for $i > 0$. We will show that the same holds for $H^{p,q}_{\overline{\partial}}$. In particular we will show that if $P = \{z = (z_1,\dots,z_n) : |z_i| < r_i \ \forall i\} \subset \mathbb{C}^n$ is a polydisc (with $r_i \in (0,\infty]$ for all i, allowed to be infinte), then $H^{p,q}_{\overline{\partial}}(P) = 0$ for all p,q with $p+q > 0$.

First we need the following generalisation of Cauchy's integral theorem for smooth functions (not necessarily holomorphic):

Theorem 1.2 (Cauchy's Integral Theorem). *Let $D = D_r(a) \subset \mathbb{C}$ be a disc, and let $f \in C^\infty(\overline{D})$ be smooth. Let $z \in D$. Then:*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w + \frac{1}{2\pi i}\int_D \frac{\partial f}{\partial \overline{w}} \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w-z}.$$

Note: Thus we see an extra term arises from the usual Cauchy integral formula if f is not holomorphic. If f is holomorphic then the extra term vanishes, since then $\frac{\partial f}{\partial \overline{w}} = 0$, and we are just left with the usual Cauchy integral formula.

Proof. Let $D_\varepsilon = D_\varepsilon(z)$, and let $\eta = \frac{1}{2\pi i} \cdot \frac{f(w)}{w-z}\,\mathrm{d}w \in \mathscr{A}^1_{\mathbb{C}}(D\backslash D_\varepsilon)$. Then

$$\mathrm{d}\eta = (\partial + \overline{\partial})\eta = \overline{\partial}\eta = -\frac{1}{2\pi i} \cdot \frac{\partial f}{\partial \overline{w}}(w) \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w-z}$$

since for the ∂ term we end up with $\mathrm{d}w \wedge \mathrm{d}w = 0$. So by Stoke's theorem:

$$(\ddagger) \qquad \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w + \frac{1}{2\pi i}\int_{D\backslash D_\varepsilon} \frac{\partial f}{\partial \overline{w}} \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w-z}$$

since $\partial(D\backslash D_\varepsilon) = \partial D - \partial D_\varepsilon$, i.e. the inner boundary has a negative orientation and so we pick up an extra sign.

We first show that $\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w \to f(z)$ as $\varepsilon \to 0$. To see this we do the usual thing and change variables to polar coordinates: set $w - z = re^{i\theta}$, so that

$$\frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\,\mathrm{d}\theta \to f(z) \qquad \text{as } \varepsilon \to 0$$

as f is smooth.

Now as $\mathrm{d}w \wedge \mathrm{d}\overline{w} = 2ir\mathrm{d}r \wedge \mathrm{d}\theta$, we see

$$\left|\frac{\partial f}{\partial \overline{w}}(w) \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w-z}\right| = 2\left|\frac{\partial f}{\partial \overline{w}} \cdot \mathrm{d}r \wedge \mathrm{d}\theta\right| \leq C|\mathrm{d}r \wedge \mathrm{d}\theta|$$

since f is smooth and so its derivative is bounded here. So hence

$$\int_{D_\varepsilon} \frac{\partial f(w)}{\partial \overline{w}} \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w-z} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

So thus as $\int_{D \backslash D_\varepsilon}(\cdots) = \int_D(\cdots) - \int_{D_\varepsilon}(\cdots)$, taking $\varepsilon \to 0$ in (‡) gives the result.

□

Theorem 1.3 ($\overline{\partial}$-Poincaré Lemma in One Variable)**.** *Let $D = D_r(a) \subset \mathbb{C}$ be a disc (can be infinite radius) and let $g \in C^\infty(\overline{D})$. Then:*

$$f(z) := \frac{1}{2\pi i}\int_D \frac{g(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\overline{w}$$

is a smooth function, i.e. $f \in C^\infty(D)$, and $\frac{\partial f}{\partial \overline{z}} = g(z)$.

Proof. We first note that we can reduce to the case where g has compact support, using a partition of unity/bump functions. So wlog assume g has compact support.

Now take $z_0 \in D$ and let $\varepsilon > 0$ be such that $D_{2\varepsilon} := D_{2\varepsilon}(z_0) \subsetneq D$. So using a partition of unity for the cover of D given by $\{D\backslash D_\varepsilon, D_{2\varepsilon}\}$, we may write

$$g(z) = g_1(z) + g_2(z)$$

where g_1 vanishes outside of $D_{2\varepsilon}$ and g_2 vanishes on D_ε (i.e. $g = \rho_1 g + \rho_2 g$ for an appropriate partition of unity ρ_1, ρ_2). In particular we see $g \equiv g_1$ on D_ε.

So define

$$f_2(z) := \frac{1}{2\pi i}\int_D \frac{g_2(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\overline{w}.$$

Then $f_2(z)$ is smooth on D_ε as g_2 vanishes on D_ε, and so for each $z \in D_\varepsilon$, g_2 vanishes near z and so we avoid the pole in the integrand. This smoothness allows us to differentiate under the integral sign, and so we see

$$\frac{\partial f_2}{\partial \overline{z}}(z) = \frac{1}{2\pi i}\int_D \underbrace{\frac{\partial}{\partial \overline{z}}\left(\frac{g_2(w)}{w-z}\right)}_{=0 \text{ as no } \overline{z} \text{ terms as holomorphic}} \mathrm{d}w \wedge \mathrm{d}\overline{w} = 0$$

(where the integrand is holomorphic here as again g_2 vanishes near the pole). Thus we see f_2 won't effect the derivative we are interested in.

Now as $g_1(z)$ has compact support, we can write

$$\frac{1}{2\pi i}\int_D \frac{g_1(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\overline{w} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\overline{w}$$

as $g_1 \equiv 0$ outside D. So setting $w - z = u$ we get

$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(u+z)}{u} \, \mathrm{d}u \wedge \mathrm{d}\overline{u} = -\frac{1}{\pi} \int_{\mathbb{C}} g_1(z + re^{i\theta}) e^{-i\theta} \, \mathrm{d}r \wedge \mathrm{d}\theta \quad =: f_1(z)$$

where we have changed to polar coordinates in the last integral. From this definition we see that f is $C^\infty(D)$, and so we can differentiate under the integral sign (differentiating the last expression w.r.t $\overline{z}$, then using the chain rule on $\frac{\partial g}{\partial \overline{z}}$ to change back to $\overline{w}$ and then working the change of variables back through) to see

$$\frac{\partial f_1}{\partial \overline{z}}(z) = \int_{\mathbb{C}} \frac{\partial g_1(w)}{\partial \overline{w}} \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z}.$$

So by Cauchy's integral formula (Theorem 1.2) we get

$$g_1(z) = \underbrace{\frac{1}{2\pi i} \int_{\partial D} \frac{g_1(w)}{w - z} \, \mathrm{d}w}_{=0 \text{ as } g_1 = 0 \text{ outside } D_\varepsilon \text{ and so in particular on } \partial D} + \overbrace{\frac{1}{2\pi i} \int_D \frac{\partial g_1(w)}{\partial \overline{w}} \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z}}^{\text{what we want above}} \quad = \frac{\partial f_1}{\partial \overline{z}}(z).$$

So setting $f = f_1 + f_2$, we get $\frac{\partial f}{\partial \overline{z}} = g_1(z) = g(z)$ for $z \in D_\varepsilon$, and f is given by the correct formula here.

So this works on $D_\varepsilon = D_\varepsilon(z_0)$ But then as $z_0 \in D$ was arbitrary, this works for every point in D (with the same formula for f as it independent of z_0 so agrees on overlaps of different $D_\varepsilon(z_0')$ when patching together) and so we are done.

□

Thus using the $\overline{\partial}$-Poincaré lemma, if $\alpha = g \mathrm{d}\overline{z} \in \mathscr{A}^{0,1}_{\mathbb{C}}(D)$ is simple, then defining f in terms of g as in the $\overline{\partial}$-Poincaré lemma we have $\overline{\partial} f = \alpha$.

To simplify things we use multi-index notation, i.e. if $I = \{I_1, \ldots, I_k\}$, then

$$\mathrm{d}z_I = \mathrm{d}z_{I_1} \wedge \cdots \wedge \mathrm{d}z_{I_k}, \quad f_I = f_{I_1,\ldots,I_k}, \quad \text{and} \quad \frac{\partial}{\partial z_I} = \frac{\partial^k}{\partial z_{I_1} \cdots \partial z_{I_k}}.$$

We write $|I| = k$ for such a multi-index I.

Lemma 1.4. *Let $U \subset \mathbb{C}^n$ be open, and let B, B' be bounded polydiscs with $B \subsetneq B' \subset U$. Then for any multi-indices I, J, $\exists$ a constant C_{IJ} such that, for all u holomorphic in U,*

$$\left\| \frac{\partial}{\partial z_I} \frac{\partial}{\partial \overline{z}_J} u \right\|_{C^0(B)} \le C_{IJ} \|u\|_{C^0(B')}$$

where $\|\cdot\|_{C^0(B)}$ is the supremum norm.

Proof. This follows from the multivariate Cauchy integral formula in the same way as the one variable case (and the multivariate Cauchy integral formula follows easily from the single variable version).

□

Lemma 1.4 simply tells us that we can bound all derivatives on a smaller ball by just u itself.

Corollary 1.1. *Let $(u_k)_k$ be a sequence of holomorphic functions on U. Suppose $u_k \to u$ locally uniformly (i.e. uniformly on all compact subsets of U). Then u is holomorphic.*

Proof. By applying Lemma 1.4 we see that all derivatives of the u_k converge uniformly (as Lemma 1.4 gives that they are uniformly Cauchy), and thus u must be smooth. But then applying Lemma 1.4 again we see that

$$\frac{\partial u_k}{\partial \overline{z}_j} \to \frac{\partial u}{\partial \overline{z}_j}$$

uniformly, and so as $\frac{\partial u_k}{\partial \overline{z}_j} = 0$ for all k, j (as the u_k are holomorphic), we see that $\frac{\partial u}{\partial \overline{z}_j} = 0$ for all j, i.e. $\overline{\partial} u = 0$. So u is holomorphic.

□

Theorem 1.4 (The $\overline{\partial}$-Poincaré Lemma (proof due to Grothendieck))**.** *Let $P = P_r(a) = \{z : |z_i - a_i| < r_i \ \forall i\} \subset \mathbb{C}^n$ be a polydisc with $r_i \in (0, \infty]$. Then for all $q > 0$, we have $H^{p,q}_{\overline{\partial}}(P) = 0$, i.e.*

$$\text{if } \overline{\partial}\omega = 0, \text{ then } \exists \psi \text{ with } \overline{\partial}\psi = \omega.$$

Proof. We first reduce to the $p = 0$ case. Indeed, if $w \in \mathscr{A}^{p,q}_{\mathbb{C}}(P)$ is closed, i.e. $\overline{\partial} w = 0$, then we may write

$$w = \sum_{|I|=p} \varphi_I \wedge \mathrm{d}z_I$$

with $\varphi_I \in \mathscr{A}^{0,q}_{\mathbb{C}}(U)$ satisfying $\overline{\partial}\varphi_I = 0$. Hence if we can find ψ_I with $\overline{\partial}\psi_I = \varphi_I$, then we would have

$$\overline{\partial}\left(\sum_{|I|=p} \psi_I \wedge \mathrm{d}z_I\right) = w$$

and so we would be done. Thus we can wlog assume $p = 0$. The proof is now in two steps. Assume $\overline{\partial} w = 0$ throughout.

> **Step 1:** Let $w \in \mathscr{A}^{0,q}_{\mathbb{C}}(P)$ be closed. We first show that if $P' = P_s(a)$ with $s < r$ (all s_i in particular are <u>finite</u>) then we can find $\psi \in \mathscr{A}^{0,q-1}_{\mathbb{C}}(P')$ with $\overline{\partial}\psi = w|_{P'}$

To see this, write $w = \sum_{|I|=q} w_I \mathrm{d}\overline{z}_I$, with the w_I smooth functions. Let us say $\boldsymbol{w \equiv 0}$ **modulo** $\mathbf{d}\boldsymbol{\overline{z}_1, \ldots, }\mathbf{d}\boldsymbol{\overline{z}_k}$ if $w_I = 0$ unless $I \subset \{1, \ldots, k\}$. We shall prove that if $w \equiv 0$ modulo $\mathrm{d}\overline{z}_1, \ldots, \mathrm{d}\overline{z}_k$, then $\exists \psi \in \mathscr{A}^{0,q-1}_{\mathbb{C}}(P')$ such that $w - \overline{\partial}\psi \equiv 0$ modulo $\mathrm{d}\overline{z}_1, \ldots, \mathrm{d}\overline{z}_{k-1}$. Then by induction (as the $k = n$ case being vacuously true for any w), this will prove Step 1.

So suppose $w \equiv 0$ modulo $\mathrm{d}\overline{z}_1, \ldots, \mathrm{d}\overline{z}_k$, and write $w = w_1 \wedge \mathrm{d}\overline{z}_k + w_2$, with $w_2 \equiv 0$ modulo $\mathrm{d}\overline{z}_1, \ldots, \mathrm{d}\overline{z}_{k-1}$ (i.e. just take all terms in w involving $\mathrm{d}\overline{z}_k$ and group them together). So we have/can write

$$w_1 = \sum_{|I|=q,\ k\in I} w_I \mathrm{d}\overline{z}_{I\setminus\{k\}}$$

as we have removed $\mathrm{d}\overline{z}_k$. Then since $\overline{\partial} w = 0$, we have

$$(\star) \qquad \frac{\partial w_I}{\partial \overline{z}_l} = 0 \qquad \text{for } l \neq k.$$

Now set $\psi = (-1)^{k-1}\sum_{I:k\in I} \psi_I \mathrm{d}\overline{z}_{I\setminus\{k\}}$, where

$$\psi_I := \frac{1}{2\pi i}\int_{|\zeta|\leq s_k} w_I(z_1,\dots,z_{k-1},\zeta,z_{k+1},\dots,z_n)\cdot \frac{\mathrm{d}\zeta\wedge \mathrm{d}\overline{\zeta}}{\zeta - z_k}.$$

Then $\frac{\partial \psi_I}{\partial \overline{z}_k} = w_I$ by the $\overline{\partial}$-Poincaré lemma in one variable, and for $l\neq k$,

$$\frac{\partial \psi_I}{\partial \overline{z}_l} = \frac{1}{2\pi i}\int_{|\zeta|\leq s_k} \frac{\partial w_I}{\partial \overline{z}_l}(z_1,\dots,z_{k-1},\zeta,z_{k+1},\dots,z_n)\cdot \frac{\mathrm{d}\zeta\wedge \mathrm{d}\overline{\zeta}}{\zeta - z_k} = 0 \qquad \text{by } (\star).$$

Hence $w - \overline{\partial}\psi = 0$ modulo $\mathrm{d}\overline{z}_1,\dots,\overline{z}_{k-1}$, since $\overline{\partial}\psi$ cancels out $w_1\wedge \mathrm{d}\overline{z}_k$ (this is why the factor of $(-1)^{k-1}$ is in the definition of ψ, since we must commute the $\mathrm{d}\overline{z}_k$ factor through $\mathrm{d}\overline{z}_{I\setminus\{k\}}$). Thus as described above, this completes the proof of Step 1.

Step 2: Remove the use of $s < r$ in Step 1.

Let $r_{j,k}$ be a strictly increasing sequence in $\mathbb{R}$ (so in particular all terms are finite) with $r_{j,k}\to r_k$ as $j\to\infty$, for all $k = 1,\dots,n$. Let $P_j = P_{r_j}(a)$. Then by Step 1, we know that we can find $\psi_j \in \mathscr{A}^{0,q-1}_{\mathbb{C}}(P_j)$ with $\overline{\partial}\psi_j = w$ on P_j.

We first prove the $q\geq 2$ case, leaving the $q = 1$ case for last. We first need to modify the ψ_j so that they are compatible with one another on overlaps. So since $\overline{\partial}\left(\psi_j - \psi_{j+1}\right) = 0$ on P_j, by Step 1 we can choose $\beta_{j+1}\in \mathscr{A}^{0,q-2}_{\mathbb{C}}(P_{j-1})$ with $\psi_j - \psi_{j+1} = \overline{\partial}\beta_{j+1}$ on $P_{j-1}\subset P_j$. Now extend all the ψ_{j+1},β_{j+1} smoothly to P in such a way such that $\beta_{j+1}\equiv 0$ outside a compact subset of P_j. Then set:

$$\varphi_{j+1} = \psi_{j+1} + \overline{\partial}\beta_{j+1}.$$

This produces a sequence $(\varphi_j)_j$ such that $\overline{\partial}\varphi_{j+1} = w$ on P_{j+1}, and $\varphi_{j+1} = \varphi_j$ on $\underline{P_{j-1}}$. To see this last equality on P_{j-1}, by construction we have

$$\varphi_j|_{P_{j-1}} - \varphi_{j+1}|_{P_{j-1}} = \overline{\partial}\beta_j|_{P_{j-1}}$$

and by construction β_j vanishes outside a compact subset of P_{j-1}. Thus we have that $\varphi_j|_{P_{j-1}}$ and $\varphi_{j+1}|_{P_{j-1}}$ agree on an open subset of P_{j-1} and so by the identity principle they must agree on all of P_{j-1}.

Thus the sequence $(\varphi_j)_j$ converges to some φ on P (due to this compatibility as the φ_j agree on the smaller polydiscs), and moreover this φ has $\overline{\partial}\varphi = w$ on P (i.e. for fixed J, for all j sufficiently large we have $\varphi|_{P_{r_J}} = \varphi_j$, and so in P_{r_J} we have $\overline{\partial}\varphi|_{P_{r_J}} = \overline{\partial}\varphi_J = w$. So taking $J\to\infty$ we get $\overline{\partial}\varphi = w$ on all of P).

Now we just need to consider the $q = 1$ case, i.e. when w is a $(0,1)$-form.

In this case the ψ_j above are just functions. We construct a sequence φ_j on P_j such that:

$$(\dagger)\qquad \begin{cases} \overline{\partial}\varphi_j = w \text{ on } P \\ \varphi_{j+1} - \varphi_j \text{ is holomorphic on } P_j \\ \|\varphi_{j+1} - \varphi_j\|_{C^0(P_{j-1})} < 2^{-j}. \end{cases}$$

Assuming this, the $(\varphi_j)_j$ converge locally uniformly to some φ on P. Moreover, $\varphi - \varphi_j$ is holomorphic on P_j, as it is the local uniform limit of $(\varphi_{j+l} - \varphi_j)_{l\geq 1}$ (using Corollary 1.1 as these are all holomorphic).

So hence $\overline{\partial}\varphi = \overline{\partial}\varphi_j = w$ on P_j (since $\overline{\partial}(\varphi - \varphi_j) = 0$) and hence $\partial\varphi = w$ on P.

So all that remains is to construct a sequence $(\varphi_j)_j$ as in ($\dagger$). We know that we can solve $\overline{\partial}\psi_j = w$ on P_j as before (using Step 1). Set $\varphi_1 = \psi_1$. We then construct φ_{j+1} inductively on j. Since $\overline{\partial}(\varphi_j - \psi_{j+1}) = 0$ on P_j, we see that $\varphi_j - \psi_{j+1}$ is holomorphic on P_j, and hence it has a Taylor series expansion valid on P_j. Truncating the Taylor series gives a polynomial γ_{j+1}, and truncating at a high enough degree gives

$$\|\varphi_j - \psi_{j+1} - \gamma_{j+1}\|_{C^0(P_{j-1})} < 2^{-j}.$$

Then extend γ_{j+1} to a holomorphic function on P_j, and set $\varphi_{j+1} := \psi_{j+1} + \gamma_{j+1}$. Then $\partial\varphi_{j+1} = w$ on P_{j+1}, $\varphi_{j+1} - \varphi_j$ is holomorphic on P_j, and

$$\|\varphi_{j+1} - \varphi_j\|_{C^0(P_{j-1})} < 2^{-j}.$$

Thus we have constructed such a sequence and thus are done.

$\square$

2. Sheaves and Cohomology

We want to compare Dolbeault cohomology with sheaf cohomology. So first we need to discuss sheaves. Let X be a topological space.

Definition 2.1. *A* ***presheaf*** $\mathscr{F}$ ***of groups*** *on* X *consists of abelian groups* $\mathscr{F}(U)$ *for all* $U \subset X$ *open, and restriction homomorphisms* $r_{VU} : \mathscr{F}(U) \to \mathscr{F}(V)$ *for all* $V \subset U$ *open, such that*

$$r_{VW} \circ r_{WU} = r_{VU} \qquad \text{and} \qquad r_{UU} = \mathrm{id}_{\mathscr{F}(U)}$$

i.e. if we restrict from U *to* W *and then* W *to* V*, this is the same as just restricting from* U *to* V*.*

One similarly defines presheaves of vector spaces. Most often $\mathscr{F}(U)$ is some class of functions on U, with restrictions given just by restricting the functions, and so in this case we write: $r_{VU}(s) \equiv s|_V$.

Another frequent example given by $\mathscr{F}(U)$ consists of sections of some vector bundle. Thus we call:

Definition 2.2. *For* $\mathscr{F}$ *a presheaf on* X*, elements of* $\mathscr{F}(U)$ *are called* ***sections****.*

Definition 2.3. *A presheaf* $\mathscr{F}$ *on* X *is a* ***sheaf*** *if in addition we have:*

(i) For all $s \in \mathscr{F}(U)$*, if* $U = \cup_i U_i$ *is an open cover and* $s|_{U_i} = 0$ *for all* i*, then* $s = 0$*.*

(ii) If $U = \cup_i U_i$ *is an open cover, and we have* $s_i \in \mathscr{F}(U_i)$ *with* $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ *for all* i, j*, then* $\exists s \in \mathscr{F}(U)$ *with* $s|_{U_i} = s_i$ *for all* i*.*

Remark: Condition (i) of a sheaf tells us that the local behaviour of a section uniquely determines its global behaviour, whilst (ii) tells us that we can build global sections from local compatible behaviours. Thus equally (i) tells us that this construction of a global section is unique.

Example 2.1. *The following are all sheaves on a complex manifold:*

(i) $C^0(U) = \{$*Continuous functions on* $U\}$.

(ii) $C^\infty(U) = \{$*Smooth functions on* $U\}$.

(iii) $\mathscr{A}^{p,q}_{\mathbb{C}}(U) = \{(p,q)$*-forms on* $U\}$.

(iv) $\mathscr{O}(U) := \{$*holomorphic functions on* $U\}$.

(v) $\mathscr{O}^*(U) := \{$*Nowhere vanishing holomorphic functions on* $U\}$.

(vi) $\Omega^p(U) := \{$*holomorphic* p*-forms on* $U\}$ $\quad \left(\equiv \{\text{sections } s \in \mathscr{A}^{p,0}_{\mathbb{C}}(U) \text{ with } \overline{\partial} s = 0\}\right)$.

All of these are naturally vector spaces except (v), which is a group with multiplication being the group action.

Definition 2.4. *A* ***morphism*** $\alpha : \mathscr{F} \to \mathscr{G}$ *of (pre-)sheaves on* X *consists of homomorphisms* $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$ *for all* $U \subset X$ *open, such that if* $V \subset U$ *is open, the diagram*

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\alpha_U} & \mathscr{G}(U) \\ \downarrow{\scriptstyle r_{VU}} & & \downarrow{\scriptstyle r_{VU}} \\ \mathscr{F}(V) & \xrightarrow{\alpha_V} & \mathscr{G}(V) \end{array}$$

commutes.

Definition 2.5. *For* $\mathscr{F}, \mathscr{G}, \mathscr{H}$ *sheaves, we say that the sequence*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \longrightarrow 0$$

is ***exact*** *if for all* $U \subset X$ *open, the sequence*

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha_U} \mathscr{G}(U) \xrightarrow{\beta_U} \mathscr{H}(U) \longrightarrow 0$$

is an exact sequence (in the usual sense of abelian groups), *<u>and</u>* *if whenever* $s \in \mathscr{H}(U)$ *and* $x \in U$, $\exists$ *a neighbourhood* V *of* x *and* $t \in \mathscr{G}(V)$ *with:* $\beta_V(t) = s|_V$.

Example 2.2 (The Exponential Short Exact Sequence)**.** *The sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2\pi i} \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \longrightarrow 0$$

is an exact sequence of sheaves, and is called the ***exponential short exact sequence****. Here* $\mathbb{Z}$ *is the constant sheaf and so* $\mathbb{Z}(U) = \{$*locally constant (continuous) functions* $U \to \mathbb{Z}\}$(ii)*, and* exp *is the exponential map, sending* $f \mapsto \exp(f)$.

The exactness of $0 \to \mathbb{Z}(U) \overset{\times 2\pi i}{\to} \mathscr{O}(U) \overset{\exp}{\to} \mathscr{O}^*(U)$ *is clear: if* $f \in \mathscr{O}^*(U)$ *is nowhere vanishing, then one can take a local branch of* log *on some* $V \subset U$ *to obtain the last condition for an exact sequence of sheaves.*

Note that it is <u>*not true*</u> *that, if* $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\} \equiv B_1(0)\backslash\{0\}$ *that*

$$0 \longrightarrow \mathbb{Z}(\Delta^*) \xrightarrow{\times 2\pi i} \mathscr{O}(\Delta^*) \xrightarrow{\exp} \mathscr{O}^*(\Delta^*) \longrightarrow 0$$

is exact. This is because including the last $\to 0$ *map can lose the exactness, essentially because we can only locally construct* log*, but not globally.*

Definition 2.6. *Let* $\mathscr{F}$ *be a sheaf on* X *and let* $x \in X$*. Then the* ***stalk*** $\mathscr{F}_x$ *of* $\mathscr{F}$ *at* x *is:*

$$\mathscr{F}_x := \frac{\{(U,s) : x \in U \subset X,\ s \in \mathscr{F}(U)\}}{\sim}$$

where $(U,s) \sim (V,t)$ *if* $\exists W \subset U \cap V$ *open with* $s|_W = t|_W$.

(ii)Similarly we can define a sheaf $\mathbb{C}$ by $\mathbb{C}(U) := \{$Continuous functions $U \to \mathbb{C}$, where $\mathbb{C}$ has the discrete topology$\}$.

So intuitively the stalk at x is all possible local behaviours about x (think of the identity principle for the equivalence relation).

Note: A morphism $\mathscr{F} \to \mathscr{G}$ induces a map of stalks $\mathscr{F}_x \to \mathscr{G}_x$ for all x.

Exercise: Show that:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0 \quad \Longleftrightarrow \quad 0 \longrightarrow \mathscr{F}_x \longrightarrow \mathscr{G}_x \longrightarrow \mathscr{H}_x \longrightarrow 0$$

is exact is exact for all $x \in X$.

Definition 2.7. *The* ***kernel*** *of $\alpha : \mathscr{F} \to \mathscr{G}$ is the sheaf defined by*

$$\ker(\alpha)(U) := \ker(\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)).$$

The definitions of cokernel and image are more complicated, as we want them to be compatible with the sheaf definitions.

2.1. Čech Cohomology.

Our aim is to define the Čech cohomology groups $\check{H}(X, \mathscr{F})$ for $\mathscr{F}$ a sheaf on X, and we will show:

$$H^{p,q}_{\bar{\partial}}(X) \cong \check{H}^q(X, \Omega^p)$$

are isomorphic in a natural way. We begin with an example.

Let X be a topological space with $X = U \cup V$, U, V open in X. Then if $s_U \in \mathscr{F}(U)$ and $s_V \in \mathscr{F}(V)$, when does there exists an $s \in \mathscr{F}(X)$ with $s|_U = s_U$, $s|_V = s_V$?

As $\mathscr{F}$ is a sheaf, by the sheaf conditions we know such an s exists $\Leftrightarrow s_U|_{U\cap V} = s_V|_{U\cap V}$. So thus we can define a map

$$\delta : \mathscr{F}(U) \oplus \mathscr{F}(V) \to \mathscr{F}(U \cap V) \quad \text{via} \quad \delta(s_U, s_V) := s_U|_{U\cap V} - s_V|_{U\cap V}.$$

Then clearly by the above discussion we have $\ker(\delta) \cong \mathscr{F}(X)$.

Now if $U = \{U_\alpha\}_\alpha$ is a locally finite open cover of X (we will need the locally finiteness later for working with partitions of unity) indexed by a subset of $\mathbb{N}$ (or any ordered set) we write:

$$U_{\alpha_0 \cdots \alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}.$$

Then we define:

$$C^0(U, \mathscr{F}) = \prod_\alpha \mathscr{F}(U_\alpha), \qquad C^1(U, \mathscr{F}) = \prod_{\alpha<\beta} \mathscr{F}(U_{\alpha\beta})$$

and in general

$$C^p(U, \mathscr{F}) = \prod_{\alpha_0 < \cdots < \alpha_p} \mathscr{F}(U_{\alpha_0 \cdots \alpha_p}).$$

Now if $\sigma \in C^p(U, \mathscr{F})$, we also define:

$$\sigma_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_p} = -\sigma_{\alpha_0 \ldots \alpha_{i+1} \alpha_i \cdots \alpha_p}.$$

We are now ready to construct a cohomology theory. As we usually do when constructing a cohomology theory, we define the boundary map $\delta : C^p(U,\mathscr{F}) \to C^{p+1}(U,\mathscr{F})$ by:

$$(\delta\sigma)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \, \sigma_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_{p+1}}\Big|_{U_{\alpha_0\cdots\alpha_{p+1}}}.$$

Example 2.3. *Suppose $U = \{U_1, U_2, U_3\}$, and $\sigma = \{\sigma_1,\sigma_2,\sigma_3\} \in C^0(U,\mathscr{F})$. Then:*

$$(\delta\sigma)_{0,1} = (-1)^0\sigma_1 + (-1)^1\sigma_0 = \sigma_1 - \sigma_0$$

and similarly

$$(\delta\sigma)_{1,2} = \sigma_2 - \sigma_1 \quad \textit{and} \quad (\delta\sigma)_{0,2} = \sigma_2 - \sigma_0.$$

Then:

$$\begin{aligned}\left(\delta^2\sigma\right)_{0,1,2} &= \sum_{j=0}^{2} (-1)^j \, (\delta\sigma)_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_2}\Big|_{U_{0,1,2}} \\ &= (\delta\sigma)_{1,2} - (\delta\sigma)_{0,2} + (\delta\sigma)_{0,1} \\ &= (\sigma_2-\sigma_1) - (\sigma_2-\sigma_0) + (\sigma_1-\sigma_0) \\ &= 0.\end{aligned}$$

Thus $\delta^2 = 0$ here, which is good for a cohomology theory!

Exercise: Show that $\delta^2 = 0$ in general.

Definition 2.8. *With respect to such an open cover $U = \{U_\alpha\}_\alpha$, we define the **Čech cohomology** by:*

$$\check{H}^q(U,\mathscr{F}) := \frac{\ker\left(\delta : C^q(U,\mathscr{F}) \to C^{q+1}(U,\mathscr{F})\right)}{\mathrm{Image}\left(\delta : c^{q-1}(U,\mathscr{F}) \to C^q(U,\mathscr{F})\right)}.$$

However this definition currently depends on the open cover U of X. To define the Čech cohomology of X, we need to remove this dependence on the cover, which we do in the standard way of a direct limit.

Definition 2.9. *We say that $\sigma \in C^p(U,\mathscr{F})$ is a **cocycle** if $\delta\sigma = 0$, and a **coboundary** if $\sigma = \delta\tau$ for some τ.*

So as usual the above cohomology groups are "cycles modulo boundaries".

Example 2.4. *Consider $X = \mathbb{P}^1$, with homogeneous coordinates $[z:w]$. Let*

$$U = \{[z:1] : z \in \mathbb{C}\} = \{w \neq 0\} \quad \textit{and} \quad V = \{[1:w] : w \in \mathbb{C}\} = \{z \neq 0\}.$$

Then clearly $U, V \cong \mathbb{C}$, and $U \cap V \cong \mathbb{C}^* \equiv \mathbb{C}\backslash\{0\}$. So let $\mathscr{U} = \{U, V\}$ be an open cover of $\mathbb{P}^1$. Then

$$C^0(\mathscr{U},\mathscr{O}) = \mathscr{O}(U) \oplus \mathscr{O}(V) \qquad \textit{and} \qquad C^1(\mathscr{U},\mathscr{O}) = \mathscr{O}(U \cap V).$$

Then $\delta : C^0(\mathscr{U}, \mathscr{O}) \to C^1(\mathscr{U}, \mathscr{O})$ *can be calculated as*

$$\delta(f, g) = f(z) - g(1/z).$$

So $\ker(\delta)$ *consists of the pairs* (f, g) *such that* $f = g =$ *constant. This is seen by writing* f, g *as power series (since they are holomorphic we can do this)*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

and so

$$f(z) = g(1/z) \iff \sum_{n=0}^{\infty} a_n z^n = \sum_{n=-\infty}^{0} b_n z^n \implies a_n = b_n = 0\ \forall n > 0 \text{ and } a_0 = b_0$$

i.e. $f = g = a_0$ *are constant.*

So we know what $\ker(\delta)$ *is. We can also see that the image of* δ *consists of all holomorphic functions on* $\mathbb{C}^*$*, again by a Laurent series argument. Thus we see that*

$$\check{H}^0(\mathscr{U}, \mathscr{O}) = \mathbb{C} \quad \text{and} \quad \check{H}^i(, \mathscr{U}, \mathscr{O}) = 0 \quad \forall i > 0.$$

We will see later that this computes $H^i(\mathbb{P}^1, \mathscr{O})$*, the Čech cohomology of* $\mathbb{P}^1$*.*

So far in our quest for Čech cohomology we have used open covers. We now remove the choice of open cover to establish the true definition. As mentioned before we do this by taking a direct limit under refinements of open covers.

Definition 2.10. *Given (locally finite) open covers* U*,* V*, we say that* V ***refines*** U *if* $\exists \varphi : \mathbb{N} \to \mathbb{N}$ *increasing such that* $\forall \beta$*,* $V_\beta \subset U_{\varphi(\beta)}$*. We write* $V \leq U$ *in this case.*

Now if $V \leq U$, we have a natural map $\rho_{VU} : C^p(U, \mathscr{F}) \to C^p(V, \mathscr{F})$ given by

$$(\rho_{VU}\sigma)_{\beta_0 \cdots \beta_p} := \left(\sigma_{\varphi(\beta_0)\cdots\varphi(\beta_p)}\right)\Big|_{V_{\beta_0\cdots\beta_p}}$$

where φ is as in the definition of a refinement. One can check that $\rho_{VU} \circ \delta = \delta \circ \rho_{VU}$, and so ρ_{VU} induces a homomorphism

$$\rho : \check{H}^q(U, \mathscr{F}) \to \check{H}^q(V, \mathscr{F}) \qquad \forall q.$$

One can also check that this map is independent of the choice of φ.

Definition 2.11. *The* ***Čech cohomology of*** $\boldsymbol{X}$ *is:*

$$H^q(X, \mathscr{F}) := \varinjlim_{U} \check{H}^q(U, \mathscr{F})$$

where $\varinjlim$ *is a direct limit (defined below).*

Note: For the genuine Čech cohomology groups we omit the " ˇ " ("check") symbol.

We quickly recall the definition of a direct limit:

Recall: If I is a partially ordered set, and G_i is an abelian group for all $i \in I$ with maps $\varphi_{ij} : G_i \to G_j$ for all $i \leq j$ such that $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$, then the **direct limit** of this system is defined to be:

$$\varinjlim_{I} G_i := \frac{\bigoplus_{i\in I} G_i}{\sim}$$

where if $g_i \in G_i$, $g_j \in G_j$, we say $g_i \sim g_j$ if $\exists k$ with $i, j \leq k$ and $\varphi_{ik}(g_i) = \varphi_{jk}(g_j)$.

Intuitively, the maps betwen the G_i bump the elements of the G_i up the poset I. This equivalence relation then says that two elements are equivalent if after being pumped up the ordering by these maps, the elements are eventually become equal.

It can be shown that the direct limit as above is also an abelian group.

Thus going back to Čech cohomology, the elements of $H^q(X, \mathscr{F})$ are represented by $[\sigma_{\alpha_0\cdots\alpha_q}] \in \check{H}^q(U, \mathscr{F})$, and equality is checked on a common refinement.

We will see that in the special case where each intersection of the U_i in an open cover is isomorphic to a polydisc, then for such "good covers" we have

$$H^q(X, \mathscr{O}) = \check{H}^q(U, \mathscr{O}).$$

Example 2.5. *$\check{H}^0(U, \mathscr{F}) = \mathscr{F}(X)$ for all U, and so $H^0(X, \mathscr{F}) = \mathscr{F}(X)$ is just the global sections.*

Example 2.6. *We will show that $H^q(X, \mathscr{A}^{r,s}_{\mathbb{C}}) = 0$ for any $q > 0$.*

To see this, let $\sigma \in H^q(X, \mathscr{A}^{r,s}_{\mathbb{C}})$ be represented by $\sigma \in C^q(\mathscr{U}, \mathscr{A}^{r,s}_{\mathbb{C}})$. Then by definition we know that $\delta\sigma = 0$.

So because we have a locally finite open cover, we can find a partition of unity $(\rho_\alpha)_\alpha$ subordinate to the cover $\mathscr{U} = \{U_\alpha\}_\alpha$. Then define:

$$\tau_{\alpha_0\cdots\alpha_{q-1}} := \sum_\beta \rho_\beta \underbrace{\sigma_{\beta\alpha_0\cdots\alpha_{q-1}}}_{\text{extend by 0 to } U_{\alpha_0\cdots\alpha_{q-1}}}.$$

So $\tau \in C^{q-1}(\mathscr{U}, \mathscr{A}^{r,s}_{\mathbb{C}})$. The general computation to show $\delta\tau = \sigma$ (thus proving the result) will be left for the second example sheet. We give a special case here to demonstrate how to prove the general case.

So as a special case suppose $\mathscr{U} = \{U, V, W\}$. Then

$$(\star) \qquad 0 = \delta\sigma = \sigma_{UV} - \sigma_{UW} + \sigma_{VW}$$

and

$$\tau_U = \rho_V\sigma_{VU} + \rho_W\sigma_{WU}, \quad \tau_V = \rho_U\sigma_{UV} + \rho_W\sigma_{WV}, \quad \tau_W = \rho_U\sigma_{UW} + \rho_V\sigma_{VW}.$$

Then:
$$\begin{aligned}(\delta\tau)_{UV} = \tau_V - \tau_U &= (\rho_U\sigma_{UV} + \rho_W\sigma_{WV}) - (\rho_V\sigma_{VU} + \rho_W\sigma_{WU})\\ &= \rho_U\sigma_{UV} + \rho_V\sigma_{UV} + \rho_W\sigma_{WV} - \rho_W\sigma_{WU} \quad \text{as } \sigma_{VU} = -\sigma_{UV}\\ &= (\rho_U + \rho_V + \rho_W)\sigma_{UV} \quad \text{using } (\star)\\ &= \sigma_{UV} \quad \text{since } \rho_U + \rho_V + \rho_W \equiv 1 \text{ as a partition of unity.}\end{aligned}$$
Thus we see (similarly for other cases) that $\delta\tau = \sigma$, and thus σ is exact. Hence we have $H^q(X, \mathscr{A}^{r,s}_{\mathbb{C}}) = 0$ for any $q > 0$.

Similarly we have $H^q(X, \mathscr{A}^k_{\mathbb{R}}) = 0$ for all $q > 0$.

Now let $\beta : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then β induces a map $C^p(U, \mathscr{F}) \to C^p(U, \mathscr{G})$ for any U. These maps commute with δ, and so induce/descend to maps on the Čech cohomology: $\beta^* : H^p(X, \mathscr{F}) \to H^p(X, \mathscr{G})$.

Now suppose $0 \to \mathscr{E} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{G}$ is exact. We want to show that we get a long exact sequence (l.e.s) on cohomology. Now because α, β are morphisms on sheaves, as above we get maps
$$\alpha^* : H^p(X, \mathscr{E}) \to H^p(X, \mathscr{F}) \quad \text{and} \quad \beta^* : H^p(X, \mathscr{F}) \to H^p(X, \mathscr{G}).$$
Now we define the **coboundary maps**
$$\delta^* : H^p(X, \mathscr{G}) \to H^{p+1}(X, \mathscr{E})$$
in the following way:

> Given $\sigma \in C^p(X, \mathscr{G})$, we can pass to a refinement V of U and find $\tau \in C^p(V, \mathscr{F})$ with $\beta(\tau) = \rho_{VU}(\sigma)$ (where ρ_{VU} are the sheaf restriction maps). Now assume $\delta\sigma = 0$. Then:
> $$\beta(\delta\tau) = \delta(\beta(\tau)) = \delta\rho_{VU}(\sigma) = \rho_{VU}(\delta\sigma) = \rho_{VU}(0) = 0.$$
> Thus we can find (by exactness) $\mu \in C^{p+1}(V, \mathscr{E})$ such that $\alpha(\mu) = \delta\tau$. Then
> $$\alpha(\delta\mu) = \delta(\alpha(\mu)) = \delta^2\tau = 0$$
> as $\delta^2 = 0$. But since α is injective by exactness, this implies $\delta\mu = 0$. Thus μ defines an element of $H^{p+1}(X, \mathscr{E})$, which is what we want. Then we define:
> $$\delta^*([\sigma]) := [\mu] \in H^{p+1}(X, \mathscr{E}).$$

This gives rise to:

Theorem 2.1. *The sequence defined above:*
$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathscr{E}) \arrow[r, "\alpha^*"] & H^0(X,\mathscr{F}) \arrow[r, "\beta^*"] & H^0(X,\mathscr{G}) \arrow[dll, "\delta^*"] \\ & H^1(X,\mathscr{E}) \arrow[r, "\alpha^*"] & H^1(X,\mathscr{F}) \arrow[r, "\beta^*"] & \cdots \end{tikzcd}$$
is exact.

Proof. We will not prove this in general - see Example Sheet 2.

For all sheaves in this course, $\exists$ arbitrarily fine open covers $\mathscr{U}$ with $0 \to \mathscr{E}(U) \to \mathscr{F}(U) \to \mathscr{G}(U) \to 0$ exact for all $U \in \mathscr{U}$. In this case the theorem is also an exercise to prove - again see Example Sheet 2.

$\square$

Now we want to relate sheaf cohomology to Dolbeault cohomology. We will prove:

Theorem 2.2 (Dolbeault's Theorem). *If X is a complex manifold, then*
$$H^{p,q}_{\overline{\partial}}(X) \;\cong\; H^q(X,\Omega^p)$$
where $\Omega^p(U) = \{\sigma \in \mathscr{A}^{p,0}(U) : \overline{\partial}\sigma = 0\}$.

We will first prove a simpler version relating de Rham cohomology to sheaf cohomology for smooth manifolds, and use ideas from that proof to establish Dolbeault's theorem.

Definition 2.12. *We say that $\mathscr{F}_1 \xrightarrow{\alpha_1} \mathscr{F}_2 \xrightarrow{\alpha_2} \cdots$ is a **complex of sheaves** if $\alpha_{i+1} \circ \alpha_i = 0$ for all i.*

*We say that a complex is **exact** if $0 \to \ker(\alpha_i) \hookrightarrow \mathscr{F}_i \xrightarrow{\alpha_i} \ker(\alpha_{i+1}) \to 0$ is a short exact sequence of sheaves for all i.*

Theorem 2.3 (de Rham's Theorem). *If X is a smooth manifold, then*
$$H^i_{\mathrm{dR}}(X,\mathbb{R}) \;\cong\; H^i(X,\mathbb{R}),$$
where in Čech cohomology $H^i(X,\mathbb{R})$ by $\mathbb{R}$ we mean the constant sheaf.

Remark: Since de Rham cohomology is isomorphic to singular cohomology, it follows that
$$H^i_{\mathrm{sing}}(X,\mathbb{R}) \cong H^i(X,\mathbb{R})$$
where $H^i_{\mathrm{sing}}(X,\mathbb{R})$ is the singular cohomology of X.

Proof. The Poincaré lemma tells us that a form which is closed in X is locally exact. Thus the Poincaré lemma tells us that the sequence
$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \xrightarrow{\ \mathrm{d}\ } \mathscr{A}^1 \xrightarrow{\ \mathrm{d}\ } \mathscr{A}^2 \xrightarrow{\ \mathrm{d}\ } \cdots$$
is exact. That is, for all p, writing $Z^p = \ker(\mathscr{A}^p \xrightarrow{\mathrm{d}} \mathscr{A}^{p+1})$ for simplicity, we have exact sequences:
$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \longrightarrow Z^1 \longrightarrow 0$$
$(\star)$
$$0 \longrightarrow Z^{p-1} \longrightarrow A^{p-1} \longrightarrow Z^p \longrightarrow 0$$

for all $p > 1$. We see before in Example 2.6 that $H^q(X, \mathscr{A}^p) = 0$ for all $p > 0$ and all $p \geq 0$. Thus the long exact sequence in cohomology associated to the top line in $(\star)$ gives:

$$\begin{aligned} H^p(X, \mathbb{R}) &\cong H^{p-1}(X, Z^1) \qquad \text{as } H^p(X, \mathscr{A}^0) = H^{p-1}(X, \mathscr{A}^1) = 0 \\ &\cong H^{p-2}(X, Z^2) \qquad \text{from lower part of } (\star) \\ &\quad\vdots \\ &\cong H^1(X, Z^{p-1}). \end{aligned}$$

Then since from the lower part of $(\star)$,

$$0 \longrightarrow H^0(X, Z^{p-1}) \longrightarrow H^0(X, \mathscr{A}^{p-1}) \xrightarrow{\mathrm{d}} H^0(X, Z^p) \longrightarrow H^1(X, Z^{p-1}) \longrightarrow 0$$

is exact (the next group up in the l.e.s is 0, hence why the far right group is 0), this gives

$$H^1(X, Z^{p-1}) \cong \frac{H^0(X, Z^p)}{\mathrm{d}(H^0(X, \mathscr{A}^{p-1}))} = \frac{Z^p(X)}{\mathrm{d}(\mathscr{A}^{p-1}(X))} =: H^p_{\mathrm{dR}}(X, \mathbb{R}).$$

The first equality here just comes from exactness of the sequence, the second comes from the fact (which we previously established) $H^0(X, \mathscr{F}) \cong \mathscr{F}(X)$ for any sheaf $\mathscr{F}$ on X, and the last is by definition of the de Rham cohomology as we have an kernel quotiented by the image.

Thus combining we have $H^p(X, \mathbb{R}) \cong H^1(X, Z^{p-1}) \cong H^p_{\mathrm{dR}}(X, \mathbb{R})$ and so we are done.

□

Proof of Dolbeault's Theorem. We work in a similar way to the proof of Dolbeault's theorem above. We have an exact complex

$$0 \longrightarrow \Omega^p \longrightarrow \mathscr{A}^{p,0}_{\mathbb{C}} \xrightarrow{\overline{\partial}} \mathscr{A}^{p,1}_{\mathbb{C}} \xrightarrow{\overline{\partial}} \cdots$$

which is exact by the $\overline{\partial}$-Poincaré lemma. Write

$$\Omega^p(U) := \{\sigma \in \mathscr{A}^{p,0}_{\mathbb{C}}(U) : \overline{\partial}\sigma = 0\}.$$

Also as before set $Z^{p,q} = \ker\left(\overline{\partial} : \mathscr{A}^{p,q}_{\mathbb{C}} \to \mathscr{A}^{p,q+1}_{\mathbb{C}}\right)$. Thus we have exact sequences:

$$0 \longrightarrow \Omega^p \longrightarrow \mathscr{A}^{p,0} \longrightarrow Z^{p,1} \longrightarrow 0$$

$$0 \longrightarrow Z^{p,q-1} \longrightarrow \mathscr{A}^{p,q-1} \longrightarrow Z^{p,q} \longrightarrow 0$$

(as every open set in X contains an open subset which is biholomorphic to a polydisc). Then once again since by Example 2.6 we have $H^i(X, \mathscr{A}^{r,s}_{\mathbb{C}}) = 0$ for all $i > 0$ and all r, s, arguing as in the proof of de Rham's theorem we have:

$$\begin{aligned} H^q(X, \Omega^p) &\cong H^{q-1}(X, Z^{p,1}) \\ &\cong \cdots \cong H^1(X, Z^{p,q-1}) \\ &\cong \frac{H^0(X, Z^{p,q})}{\overline{\partial}\left(H^0\left(X, \mathscr{A}^{p,q-1}_{\mathbb{C}}\right)\right)} \\ &= \frac{Z^{p,q}(X)}{\overline{\partial}\left(\mathscr{A}^{p,q-1}_{\mathbb{C}}(X)\right)} =: H^{p,q}_{\overline{\partial}}(X) \end{aligned}$$

which proves the result. □

Remark: Note that in the last string is isomorphisms in the proof of Dolbeault's theorem we see in particular that $H^1(U_{\alpha_0\cdots\alpha_s}\ ,\ Z^{0,q-1}) = H^{0,q}_{\overline{\partial}}(U_{\alpha_0\cdots\alpha_s})$ for any $\alpha_0,\ldots,\alpha_s$ for any open cover U.

This enables us to prove one way of calculating the Čech cohomology, which is by finding a "nice" open cover:

Theorem 2.4. *Let X be a complex manifold. Suppose U is an open cover with the property that:*

$$H^p\left(U_{\alpha_0\cdots\alpha_s}\ ,\ \mathscr{O}\right) = 0 \quad \forall p \geq 1 \text{ and all } \alpha_0,\ldots,\alpha_s.$$

Then $H^p(X,\mathscr{O}) \cong H^p(U,\mathscr{O})$.

Proof. We have from the above remark:

$$H^1(U_{\alpha_0\cdots\alpha_s}, Z^{0,q-1}) = H^{0,q}_{\overline{\partial}}(U_{\alpha_0\cdots\alpha_s}) = H^q(U_{\alpha_0\cdots\alpha_s},\mathscr{O}) = 0 \quad \text{by hypothesis.}$$

Thus we see

$$0 \longrightarrow Z^{0,q-1}(U_{\alpha_0\cdots\alpha_s}) \longrightarrow \mathscr{A}^{0,q-1}_{\mathbb{C}}(U_{\alpha_0\cdots\alpha_s}) \longrightarrow Z^{0,q}(U_{\alpha_0\cdots\alpha_s}) \longrightarrow 0$$

is exact. This is true for all intersections, and so we get a short exact sequence:

$$0 \longrightarrow C^p(U,Z^{0,q-1}) \longrightarrow C^p(U,\mathscr{A}^{0,q-1}_{\mathbb{C}}) \longrightarrow C^p(U,Z^{0,q}) \longrightarrow 0\ .$$

Then considering the associated long exact sequence, and using that $\check{H}^p(U,\mathscr{A}^{0,q}) = 0$ (from Example 2.6) gives that, for all $p,q \geq 1$,

$$\check{H}^p(U,Z^{0,q}) \cong \check{H}^{p+1}(U,Z^{0,q-1}).$$

Then arguing as before in de Rham's/Dolbeault's theorem:

$$\check{H}^p(U,\mathscr{O}) = \check{H}^p(U,Z^{0,0}) \cong \check{H}^{p-1}(U,Z^{0,1}) \cong \cdots \cong \check{H}^1(U,Z^{0,p-1})$$

and also

$$\check{H}^1(U,Z^{0,p-1}) \cong \frac{Z^{0,p}(X)}{\overline{\partial}\left(\mathscr{A}^{0,p-1}(X)\right)} = H^{0,p}_{\overline{\partial}}(X) = H^p(X,\mathscr{O})$$

as required.

□

Remark: With the same hypotheses, this also shows that $H^q(X,\Omega^p) \cong \check{H}^q(U,\Omega^p)$.

Example 2.7. *Thus we see $H^q(\mathbb{C}^n,\mathscr{O}) \cong H^{0,q}(\mathbb{C}^n) = 0$ for all $q \geq 1$.*

Remark: One can also show that if $\check{H}^p(U_\alpha,\mathscr{O}) = 0$ for all $U_\alpha \in U$ (with no assumptions on the higher order intersections), then in fact $H^p(X,\mathscr{O}) \cong \check{H}^p(U,\mathscr{O})$ [see Voisin, §4 for a discussion of this]. So in particular if X is projective, then one can take U to be a cover by affine subvarities. When X is not projective, one can take a cover by **Stein manifolds** (which can be thought of as the "complex manifold version of affine varieties").

Remark: We can also show $H^p(X,\mathbb{Z}) \cong H^p_{\text{sing}}(X,\mathbb{Z})$ for the integral cohomologies.

Remark: Just as some motivation for when you might use sheaf cohomology, one usually cares about $H^0(X,\mathscr{F})$, and the higher H^i are viewed as obstructions (e.g. in the short exact sequences - like in Mittag-Leffler).

Another reason to care about H^i is the **Euler characteristic**, defined via

$$\chi(X,\mathscr{F}) := \sum_i (-1)^i \dim\big(H^i(X,\mathscr{F})\big)$$

which is additive in s.e.s's and usually constant in families (whilst H^0 is not). H^1 is also "geometric".

Now we move on from sheaf theory and cohomology theory and look at vector bundles.

3. Holomorphic Vector Bundles

Definition 3.1. *Let X be a complex manifold. Then a **holomorphic vector bundle of rank r** on X is a complex manifold E with a (holomorphic, surjective) map $\pi : E \to X$ with all fibres $\pi^{-1}(x) =: E_x$ being r-dimensional (complex) vector spaces, such that $\exists$ an open cover $\{U_\alpha\}_\alpha$ of X and biholomorphic maps $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{C}^r$ isomorphisms which commuting with the projections to X, U_α such that the induced maps on $\pi^{-1}(\{x\}) \cong \mathbb{C}^r$ are $\mathbb{C}$-linear for all $x \in X$.*

Thus this is essentially just the definition of a smooth vector bundle on a real smooth manifold, except we require the local trivialisation to be biholomorphic instead of diffeomorphic. Note that the conditions on π being holomorphic and surjective are implied from the other conditions (e.g. surjective as $\pi^{-1}(x) \cong \mathbb{C}^r \neq \emptyset$) and so we can choose to leave them out of the definition if we wish.

Definition 3.2. *A **holomorphic line bundle** is a holomorphic vector bundle of rank 1.*

Any holomorphic vector bundle induces a complex vector bundle, but not vice versa.

Definition 3.3. *Let $\pi_E : E \to X$, $\pi_F : F \to X$ be holomorphic vector bundles. Then a **morphism** $f : E \to F$ is a holomorphic map such that:*

- *(i)* $\pi_F \circ f = f \circ \pi_E$
- *(ii) The induced map $f_x : E_x \to F_x$ is linear for all $x \in X$*
- *(iii)* $\mathrm{rank}(f_x)$ *is constant with x.*

*A morphism is an **isomorphism** if f_x is an isomorphism for all $x \in X$.*

Remark: In differential geometry one usually does not require condition (iii) on a morphism. We include it to enable us to take kernels and cokernels and still end up with vector bundles.

For a holomorphic vector bundle E, its transition functions $\varphi_{\alpha\beta} \equiv \varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$ can be seen as holomorphic maps

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C})$$

i.e. $x \mapsto \varphi_{\alpha\beta}(x, \cdot)$ and $\varphi_{\alpha\beta}(x, \cdot) : \mathbb{C}^r \to \mathbb{C}^r$ is linear. These maps satisfy the usual cocycle conditions:

$$\varphi_{\alpha\alpha} = \mathrm{id}_{U_\alpha}, \qquad \varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}, \qquad \varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = \mathrm{id}_{U_\alpha \cap U_\beta \cap U_\gamma}.$$

Proposition 3.1 (Equivalence of Cocycle Data and Holomorphic Vector Bundles)**.** *Given any open cover $X = \bigcup_\alpha U_\alpha$ and holomorphic maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C})$ satisfying the cocycle conditions, there is a holomorphic vector bundle with these maps as its transition functions.*

Proof. The same as in differential geometry. □

So given a holomorphic vector bundle E and a trivialising cover $U = \{U_\alpha\}_\alpha$ of X with trivialisation maps $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$, the transition functions $\{\varphi_{\alpha\beta}\}_{\alpha,\beta} \subset C^1(U, \mathrm{GL}_r(\mathbb{C}))$, i.e. are C^1 maps $U \to \mathrm{GL}_r(\mathbb{C})$ (as usual $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$), and they satisfy the cocycle conditions as explained above. Hence by definition of the boundary map we have $\delta(\{\varphi_{\alpha\beta}\}) = 0$, and so we obtain an element of $H^1(X, \mathrm{GL}_r(\mathbb{C}))$, which we denote $[\varphi E]$ (here we are viewing $\mathrm{GL}_r(\mathbb{C})$ as a group under multiplication).

We now specialise to the case of line bundles, i.e. $r = 1$, and so $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}\backslash\{0\}$ (so invertible we do not have 0), and thus

$$H^1(X, \mathrm{GL}_1(\mathbb{C})) \cong H^1(X, \mathcal{O}^*)$$

since from before, $\mathrm{GL}_r(\mathbb{C})$ is the sheaf defined by:

$$(\mathrm{GL}_r(\mathbb{C}))(U) := \{\text{holomorphic maps } U \to \mathrm{GL}_r(\mathbb{C})\}.$$

Proposition 3.2. *There is a canonical bijection:*

$$\{\textit{holomorphic line bundles on } X \textit{ up to isomorphism}\} \longleftrightarrow H^1(X, \mathcal{O}^*).$$

Proof. We have already constructed above maps in each direction. So we need to show that these maps are inverses and the first map is well-defined (i.e. independent of the representative in the equivalence class).

Suppose $L \cong F$ are isomorphic line bundles. Choose a cover $U = \{U_\alpha\}_\alpha$ trivialising both L, F. Then we have maps:

$$\varphi_\alpha : L|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{C} \qquad \text{and} \qquad \sigma_\alpha : F|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{C}$$

giving transition maps $\varphi_{\alpha\beta}, \sigma_{\alpha\beta}$ as before. Now as L, F are isomorphic we have an isomorphism $f : L \to F$ giving maps $f_\alpha : L|_{U_\alpha} \to F|_{U_\alpha}$. Now define:

$$h_\alpha := \sigma_\alpha \circ f_\alpha \circ \varphi_\alpha^{-1}.$$

Then $h_\alpha : U_\alpha \times \mathbb{C} \to U_\alpha \times \mathbb{C}$, or can view it as a section of $\mathcal{O}^*$. Moreover,

$$\begin{aligned}
(\delta h)_{\alpha\beta} = h_\alpha h_\beta^{-1} &= \left(\sigma_\alpha f_\alpha \varphi_\alpha^{-1}\right)\left(\varphi_\beta f_\beta^{-1} \sigma_\beta^{-1}\right) \\
&= \sigma_\alpha f_\alpha \varphi_{\beta\alpha} f_\beta^{-1} \sigma_\beta^{-1} \\
&= \sigma_\alpha \varphi_{\beta\alpha} f_\alpha f_\beta^{-1} \sigma_\beta^{-1} && \text{by definition of an isomorphism} \\
&= \sigma_{\alpha\beta} \varphi_{\alpha\beta}^{-1} && \text{as } f_\alpha f_\beta^{-1} = \mathrm{id}
\end{aligned}$$

and thus the transition maps give the same element of Čech cohomology (as δh is exact), i.e. $[\sigma] = [\varphi] \in H^1(X, \mathcal{O}^*)$.

Thus the first map is well-defined.

For the converse, suppose L, F are line bundles with $[\varphi] = [\sigma] \in H^1(X, \mathcal{O}^*)$. Thus we can find $h = \{h_\alpha\}_\alpha \in C^0(U, \mathcal{O}^*)$ with $(\delta h)_{\alpha\beta} = \varphi_{\alpha\beta}^{-1} \sigma_{\alpha\beta}$ (as $[\varphi^{-1}\sigma]$ is exact). Now let $f_\alpha : L|_{U_\alpha} \to F|_{U_\alpha}$ be given by:

$$f_\alpha := \sigma_\alpha^{-1} h_\alpha \varphi_\alpha$$

(i.e. just mimicking what we did above). We claim that the f_α induce a map $f : L \to M$, i.e. $f_\alpha \circ f_\beta^{-1} = \mathrm{id}$ on $U_\alpha \cap U_\beta$ (so we can patch the f_α together to get a global map). But indeed,

$$f_\alpha f_\beta^{-1} = \sigma_\alpha^{-1} h_\alpha \varphi_\alpha \varphi_\beta^{-1} h_\beta^{-1} \sigma_\beta = \cdots = \mathrm{id}$$

as in the above calculation. Thus we are done.

□

Remark: A similar result is true for all ranks with the right definition of Čech cohomology for sheaves of (non-abelian) groups. For line bundles all groups are abelian and so we can use the sheaf theory we looked at.

Definition 3.4. *The **Picard group** is:*

$$\mathrm{Pic}(X) := \{\textit{line bundles on } X \textit{ up to isomorphism}\}.$$

Proposition 3.3. $\mathrm{Pic}(X)$ *is a group, with the group action being the tensor product of line bundles, and* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$.

Proof. The easiest way of doing this is using the transition functions. The transition functions for $L \otimes F$ are $\varphi_{\alpha\beta} \otimes \sigma_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. So if L^* is the dual line bundle of a line bundle L, we have

$$L \otimes L^* \cong \mathcal{O} \qquad \text{and} \qquad L \otimes \mathcal{O} \cong L$$

i.e. L^* is the inverse of L and $\mathcal{O}$ is the identity element. So $\mathrm{Pic}(X)$ is a group, and from this construction via line bundles (using Proposition 3.2) we see $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$.

□

Example 3.1. *Any linear algebra operation gives an operation on vector bundles, e.g.:*

(i) $E \oplus F$ *is a vector bundle with transition functions* $\begin{pmatrix} \varphi_{\alpha\beta} & 0 \\ 0 & \sigma_{\alpha\beta} \end{pmatrix}$.

(ii) $E \otimes F$ *is a vector bundle with transition functions* $\varphi_{\alpha\beta} \otimes \sigma_{\alpha\beta} \in \mathrm{GL}(\mathbb{C}^r \otimes \mathbb{C}^{r'}) \equiv \mathrm{GL}_{r+r'}(\mathbb{C})$.

(iii) $\Lambda^k E$ *is a vector bundle with transition functions* $\Lambda^k \varphi_{\alpha\beta}$.

If $k = r$ *we write* $\Lambda^r E =: \det(E)$*, the **determinant line bundle**. Thus to any vector bundle we get an associated line bundle via the top exterior power, i.e. the determinant line bundle.*

Definition 3.5. *A* ***holomorphic section*** *s of a holomorphic vector bundle E over $U \subset X$ is a holomorphic map $s : U \to E$ with $\pi \circ s = \mathrm{id}$. We write $\mathcal{O}(E)$ for the sheaf of holomorphic sections of E, i.e.*

$$\mathcal{O}(E)(U) := \{\text{holomorphic sections of } E \text{ over } U \subset X\}.$$

Then the usual $\mathcal{O}$ can be seen as the sheaf of sections of the trivial line bundle $X \times \mathbb{C}$.

Definition 3.6. *If $\mathcal{F}, \mathcal{G}$ are sheaves, a morphism of sheaves φ is an* ***isomorphism*** *if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all $U \subset X$.*

Definition 3.7. *A sheaf $\mathcal{F}$ is* ***locally free*** *of rank r if $\forall x \in X$, $\exists$ open $U \subset X$, $x \in U$, with*

$$\mathcal{F}|_U \cong \underbrace{(\mathcal{O} \oplus \cdots \oplus \mathcal{O})}_{r \text{ copies}}|_U.$$

Remark: We never actually defined a **restriction sheaf**, so we quickly note the definition here: if $\mathcal{F}$ is a sheaf on X and $U \subset X$ is open, then $\mathcal{F}|_U$ is a sheaf with for all $V \subset U$ open, $\mathcal{F}|_U(V) := \mathcal{F}(V)$.

Proposition 3.4. *Associating to a holomorphic vector bundle its sheaf of sections gives a canonical bijection:*

$$\{\text{holomorphic vector bundles up to isomorphism}\} \longleftrightarrow \{\text{locally free sheaves up to isomorphism}\}.$$

Proof. Clearly the sheaf of sections of a holomorphic vector bundle E is locally free, as E is locally isomorphic to $U_\alpha \times \mathbb{C}^r$. So this is a map in one direction.

For the converse, if we have trivialisations $\varphi_\alpha : \mathcal{F}|_{U_\alpha} \xrightarrow{\cong} \mathcal{O}^{\oplus r}\big|_{U_\alpha}$ (by definition of a locally free sheaf), then the transition maps

$$\varphi_{\alpha\beta} \equiv \varphi_\alpha \circ \varphi_\beta^{-1} : \mathcal{O}^{\oplus r}(U_\alpha \cap U_\beta) \xrightarrow{\cong} \mathcal{O}^{\oplus r}(U_\alpha \cap U_\beta)$$

are given by a matrix of holomorphic functions on $U_\alpha \cap U_\beta$, giving the cocycle conditions and hence a holomorphic vector bundle. So this gives a map from locally free sheaves to holomorphic vector bundles.

But we now need to check that these maps are inverses of one another. But this is straightforward from how we construct vector bundles from cocycle data/local trivialisations.

$\square$

Notation: We write for E a holomorphic vector bundle,

$$H^i(X, E) := H^i(X, \mathcal{O}(E)).$$

Example 3.2. *Recall that $TX^{1,0}$ is the holomorphic tangent bundle. We want to show that this is actually a holomorphic vector bundle. So we need to show that the transition functins are actually holomorphic.*

So we need to remember how we actually constructed $T^{(1,0)}X \equiv TX^{1,0}$. Let $X = \bigcup_\alpha U_\alpha$ be an open cover by charts, and $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$. The Jacobian of the transition maps $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is

$$J(\varphi_{\alpha\beta}) = \left(\frac{\partial^\gamma \varphi_{\alpha\beta}}{\partial z^\delta} \left(\varphi_{\alpha\beta}(z)\right) \right)_{\gamma,\delta}.$$

Then by Example Sheet 1, Q1, we know $T^{(1,0)}X$ has transition functions $\varphi_{\alpha\beta}(z) := J(\varphi_{\alpha\beta})(\varphi_\beta(z))$. So as we can view $\varphi_{\alpha\beta}(z) \in \mathrm{GL}_n(U_\alpha \cap U_\beta)$, these are holomorphic. □

Now as always whenever we have a vector bundle we get a line bundle via the top exterior power (we will see that line bundles are somewhat "more fundamental" than vector bundles). Since we always have a holomorphic vector bundle on a complex manifold (via) the holomorphic tangent bundle, this means every complex manifold has a canonical line bundle (which turns out to be the only natural line bundle on a complex manifold).

Definition 3.8. *We define the* **canonical line bundle** *of a complex manifold X by:*

$$K_X := \det\left(T^*X^{1,0}\right) \equiv \Lambda^n \left(T^*X^{1,0}\right)$$

where $T^*X^{1,0} \cong (TX^{1,0})^*$. This is a holomorphic line bundle.

Another key example of line bundles is the canonical line bundle on $\mathbb{P}^n$:

Example 3.3. *Here we construct line bundles in $\mathbb{P}^n$. Each point $l \in \mathbb{P}^n$ corresponds to a line through 0. So consider the set:*

$$\mathcal{O}(-1) := \{(l,z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : z \in l\}$$

i.e. the fibre at $l \in \mathbb{P}^n$ is just the line l in $\mathbb{C}^{n+1}$. We claim that this is a holomorphic line bundle $\mathcal{O}(-1) \to \mathbb{P}^n$.

Indeed, consider the standard cover $\mathbb{P}^n = \bigcup_{\alpha=0}^n U_\alpha$. A trivialisation of $\mathcal{O}(-1)$ over U_α is cover by:

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C} \qquad \text{sending} \qquad (l,z) \mapsto (l, z_\alpha)$$

for $z = (z_0, \ldots, z_n)$ (i.e. z_α is α'th-coordinate of z). The transition functions are then $\psi_{\alpha\beta}(l) : \mathbb{C} \to \mathbb{C}$ sending $z \mapsto \frac{l_\alpha}{l_\beta} z$, where $l = [l_0 : \cdots : l_n]$, which is holomorphic since it is linear.

Finally we need to check that $\mathcal{O}(-1)$ is a complex manifold. If $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ is as above, then define charts $\hat{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \underbrace{\mathbb{C} \times \mathbb{C}^n}_{\cong \mathbb{C}^{n+1}}$ via: $\hat{\varphi}_\alpha := (\varphi_\alpha \times \mathrm{id}) \circ \psi_\alpha$. Then one can check that this works, and so we are done. □

Definition 3.9. *$\mathcal{O}(-1)$ is called the* ***tautological line bundle*** *of $\mathbb{P}^n$.*

So we have seen that $\mathcal{O}(-1)$ is indeed a holomorphic line bundle. We then define:

$$\mathcal{O}(1) := \mathcal{O}(-1)^*$$

which is called the **hyperplane line bundle**. We then define:

$$\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}, \qquad \mathcal{O}(-k) := \mathcal{O}(-1)^{\otimes k}, \qquad \mathcal{O}(0) = \mathcal{O}$$

for all $k > 0$. We will later show that these are <u>all</u> the holomorphic line bundles on $\mathbb{P}^n$, and so

$$\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z} \qquad \text{with generator } \mathcal{O}(1).$$

Now one last important example before moving on:

Example 3.4. *If $p : Y \to X$ is a morphism and $E \to X$ is a holomorphic vector bundle, then one obtains the* ***pullback bundle*** *over Y, $p^*E \to Y$, by pulling back the transition functions via p.*

If $Y \subset X$ is a submanifold, we write $E|_Y$ for the pullback bundle of E under the inclusion map $Y \hookrightarrow X$.

Also for any projective X, we know X is biholomorphic to a subset of $\mathbb{P}^n$ (for some n), i.e. $X \subset \mathbb{P}^n$, and then X has a natural line bundle via pulling back the hyperplane line bundle, i.e. $\mathcal{O}(1)|_X \to X$.

3.1. **(Commutative) Algebra of Complex Manifolds.**

We now relate sections of line bundles, codimension one submanifolds, and meromorphic functions to one another. By the implicit function theorem, a subset $Y \subset X$ is a closed submanifold if and only if for all $p \in X$, $\exists$ a neighbourhood $U \subset X$ of p and holomorphic functions $f_1, \dots, f_k : U \to \mathbb{C}$ such that 0 is a regular value of $f = (f_1 \circ \varphi^{-1}, \dots, f_k \circ \varphi^{-1}) : \varphi(U) \to \mathbb{C}^k$, where $\varphi : U \to \mathbb{C}^n$. In this case we have

$$Y \cap U = \bigcap_{i=1}^{k} f_i^{-1}(0)$$

i.e. this is just saying that Y is locally the region cut out by some holomorphic functions.

Recall: If $U \subset \mathbb{C}^n$ is open and $f : U \to \mathbb{C}^k$ is holomorphic, then setting

$$J(f)(z) := \left(\frac{\partial f_\alpha}{\partial z_\beta}(z)\right)_{1 \le \alpha \le k,\ 1 \le \beta \le n}$$

then $z \in U$ is a **regular point** if $J(f)(z)$ is surjective. Moreover if every $z \in f^{-1}(w)$ is regular, then w is called a **regular value**.

Definition 3.10. *Let X be a complex manifold. Then an* ***analytic subvariety*** *of X is a closed subset $Y \subset X$ such that for all $p \in X$, $\exists$ a neighbourhood $U \subset X$ of p and holomorphic functions $f_1, \dots, f_k$ with $Y \cap U = \bigcap_{i=0}^{k} f_i^{-1}(0)$.*

Remark: Note that we assume no extra structure on an analytic subvariety - it is just a closed subset defined in this way. Thus the only difference between it and the above closed submanifold discussion is that in the definition of the analytic subvariety we do not assume 0 is a regular value. Thus an analytic subvariety really is just a closed subset which is locally cut out by holomorphic functions.

Definition 3.11. *For Y an analytic subvariety of X, we say $y \in Y$ is a* ***regular*** *(or* ***smooth) point*** *if one can choose the f_i in Definition 3.10 such that 0 is regular.*

Then by the implicit function theorem, if Y^S denotes the points of Y which are not regular (i.e. "S" for "singular"), then $Y^* := Y\backslash Y^S$ is naturally a complex manifold (or at least its connected components are).

Definition 3.12. *An analytic subvariety Y is* ***irreducible*** *if it cannot be written as $Y = Y_1 \cup Y_2$, with Y_1, Y_2 analytic subvarieties with $Y_1, Y_2 \neq Y$.*

Example 3.5. *The set $\left\{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\right\} \subset \mathbb{C}^2$, i.e. the union of the coordinate axes, is an analytic subvariety which is reducible, since $\{z_1 = 0\} \cup \{z_2 = 0\} = \{z_1 z_2 = 0\}$. However it is not a complex manifold since it is singular at the origin.*

Definition 3.13. *We define the* ***dimension*** *of an irreducible analytic subvariety Y to be:*

$$\dim(Y) := \dim(Y^*)$$

where the latter is well-defined as it is a complex manifold.

Similarly if Y is reducible and each irreducible component of Y has the same dimension we can define $\dim(Y)$.

Definition 3.14. *If $\operatorname{codim}(Y) = 1$, then we say Y is an* ***analytic hypersurface****.*

Now if $\mathscr{F}$ is a sheaf on X, for $x \in X$ we define $\mathscr{F}_x$ to be the stalk of $\mathscr{F}$ at x. On $\mathbb{C}^n$, we set $\mathscr{O}_{\mathbb{C}^n}$ to be the sheaf of holomorphic functions, and we set

$$\mathscr{O}_n := \mathscr{O}_{\mathbb{C}^n,0}$$

to be the stalk at $0 \in \mathbb{C}^n$. Elements of $\mathscr{O}_n$ are of the form (U, f), for $f \in \mathscr{O}_{\mathbb{C}^n}(U)$, $0 \in U$, and we have $(U, f) = (V, g)$ if $\exists$ an open set $W \subset U \cap V$ with $f|_W = g|_W$.

In the case of X an n-dimensional complex manifold, we write $\mathscr{O}_X$ for the sheaf of holomorphic functions on X. Then since X locally looks likes $\mathbb{C}^n$ we have

$$\mathscr{O}_{X,x} \cong \mathscr{O}_n$$

for any $x \in X$.

Definition 3.15. *We call elements of $\mathcal{O}_{X,x}$ **germs** of holomorphic functions.*

Note that $\mathcal{O}_n$ is a local ring, in the sense it has a unique maximal ideal, namely $\{f : f(0) = 0\}$. Functions not vanishing at 0 are invertible, and so these are the units of the ring.

We now state several results about $\mathcal{O}_n$, proved using commutative algebra and complex analysis. We shall not prove them - see Huybrechts Chapter 1 if interested.

Theorem 3.1. *$\mathcal{O}_n$ is a unique factorisation domain (UFD).*

Proof. See Huybrechts Chapter 1. □

Recall that $f \in \mathcal{O}_n$ is irreducible if f cannot be written as a product of two non-units in $\mathcal{O}_n$. Thus $\mathcal{O}_n$ being a UFD means that every element of $\mathcal{O}_n$ has a unique expression as a product of irreducible elements, up to multiplication by units.

Theorem 3.2 (Weak Nullstellensatz)**.** *Let $f, g \in \mathcal{O}_n$ with f irreducible and let U be a neighbourhood on which both f, g are defined. Suppose $\{f = 0\} \cap U \subset \{g = 0\} \cap U$. Then f divides g in $\mathcal{O}_n$, i.e. g/f is holomorphic near 0.*

Proof. See Huybrechts Chapter 1. □

Definition 3.16. *Let $U \subset \mathbb{C}^n$ be open. We shall call a set $V \subset U$ **thin** if V is locally contained in the vanishing set of a set of holomorphic functions.*

Theorem 3.3. *We have the following:*

(i) Suppose $f \in \mathcal{O}_n$ is irreducible. Then $\exists$ a thin set of codimension ≥ 2 and an open set U such that $f \in \mathcal{O}_p \equiv \mathcal{O}_{\mathbb{C}^n,p}$ is irreducible for all $p \in U \backslash V$.

(ii) If $f, g \in \mathcal{O}_n$ are coprime, then $\exists U$ and thin $V \subset U$ with f, g being coprime in $\mathcal{O}_p$ for all $p \in U \backslash V$.

Proof. See Huybrechts Chapter 1. □

Remark: Huybrechts Proposition 1.1.35 claims that one can take $V = \emptyset$, but this is wrong (as demonstrated by the counterexample $\{y^2 - xz^3 = 0\} \subset \mathbb{C}^3$, which is irreducible at $0 \in \mathbb{C}^3$ but not at any $(x_0, 0, 0)$ for x_0 near 0). The proof in Huybrechts actually proves Theorem 3.3.

So let X be a complex manifold and $Y \subset X$ an analytic hypersurface. Then if $p \in Y$, $\exists$ an open $U \ni p$ with $U \subset X$ and $\exists\, f \in \mathcal{O}_X(U)$ with $U \cap Y = f^{-1}(0) \cap U$.

Definition 3.17. *Such an f is called a **local defining equation** for Y.*

If $f = f_1 \cdots f_r$ is such that the f_i are irreducible, and also $f = g_1 \cdots g_m$ (again irreducible), then by the weak nullstellensatz (and/or since $\mathcal{O}_n$ is a UFD), after reordering and multiplying by units, we have $r = m$ and $f_i = g_i$ for all i.

Now one more result we will state without proof:

Theorem 3.4. *Let Y be an analytic hypersurface. Then Y^* is an open dense subset of Y, and*

$$Y^* \text{ is connected} \iff Y \text{ is irreducible.}$$

Moreover Y^S is contained in an analytic subvariety (of X) of codimension ≥ 2.

Proof. None given. □

3.2. **Meromorphic Functions and Divisors.**

Definition 3.18. *Let X be a complex manifold, and $U \subset X$ open. Then a **meromorphic function** on U is a map $f : U \to \amalg_{p \in U} K_p$, where K_p is the field of fractions of $\mathcal{O}_p$, such that $\forall p \in U$, $\exists$ a neighbourhood $V \subset U$ of p and $g, h \in \mathcal{O}_X(V)$ with*

$$f_q = \frac{g}{h} \qquad \forall q \in V.$$

*We denote by $\boldsymbol{K}$ the corresponding **sheaf of meromorphic functions**.*

Remark: $f(p) \in K_p$ should be implied by the definition.

Note: This is different from the definition of a holomorphic function on a Riemann surface, which is usually just an analytic map to $\mathbb{P}^1$. This however doesn't generalise well, so hence the need for the above definition.

Thus elements of K_p are of the form g/h, for $g, h \in \mathcal{O}_p$, with $h \neq 0$. We then as usual write $\boldsymbol{K}^*$ for the **sheaf of meromorphic functions which are not identically zero**.

Equivalently one can define a meromorphic function via specifying $f|_{U_\alpha} = g_\alpha/h_\alpha$, with $g_\alpha, h_\alpha \in \mathcal{O}(U_\alpha)$ [**Exercise** to show]. A meromorphic "function" is undefined (even as ∞) when both $g(p) = h(p) = 0$.

Definition 3.19. *Let $Y \subset X$ be an analytic hypersurface, $p \in Y$ regular, and f a local defining function for Y at p. Then for $g \in \mathcal{O}_{X,p}$ we define the* ***order of g along Y at p*** *to be:*

$$\mathrm{ord}_{Y,p}(g) := \max\{a \in \mathbb{N} : f^a \text{ divides } g \text{ in } \mathcal{O}_{X,p}\}.$$

This order is well-defined as $\mathcal{O}_{X,p}$ is a UFD, and the order is always finite.

Lemma 3.1. *$\exists$ a neighbourhood U of p and a thin set V of codimension ≥ 2 such that if $q \in (U \backslash V) \cap Y$, then $\mathrm{ord}_{Y,p}(g) = \mathrm{ord}_{Y,q}(g)$*

i.e. the order is locally constant up to a set of codimension ≥ 2.

Proof. Simply use Theorem 3.3(i). □

Definition 3.20. *We define the* ***order of g along Y*** *(with Y irreducible) to be*

$$\mathrm{ord}_Y(g) := \mathrm{ord}_{Y,p}(g)$$

for any $p \in Y^*$ away from the thin set found in Lemma 3.1.

Note: To define the order of g along Y we are using that Y^* is open (and so has codimension 0) and that V has codimension 2 in X, so such a p does exist.

Then one can show that if g, h are holomorphic around p, then

$$\mathrm{ord}_Y(gh) = \mathrm{ord}_Y(g) + \mathrm{ord}_Y(h).$$

This allows us to define the order of meromorphic functions:

Definition 3.21. *Let X be a complex manifold and $f \not\equiv 0$ a meromorphic function. Let Y be an irreducible analytic hypersurface of X. Then we define the* ***order of f along Y*** *by*

$$\mathrm{ord}_Y(f) := \mathrm{ord}_Y(g) - \mathrm{ord}_Y(h)$$

where $f = g/h$ at some regular point of Y.

Note that this is well-defined by the additivity of the order.

Definition 3.22. *For X a complex manifold, $f \not\equiv 0$ a meromorphic function and Y an irreducible analytic hypersurface of X, we say:*

- *f has a* ***zero of order d along Y*** *if $d = \mathrm{ord}_Y(f) > 0$*
- *f has a* ***pole of order $-d$ along Y*** *if $d = \mathrm{ord}_Y(f) < 0$.*

Definition 3.23. *A **divisor** on a complex manifold X is a formal sum*
$$D = \sum_{\alpha} a_\alpha Y_\alpha$$
where $\alpha \in \mathbb{Z}$ and the Y_α are irreducible analytic hypersurfaces, such that D is locally finite, i.e. for all $x \in X$, $\exists$ a neighbourhood $V \subset X$ of x with $Y_\alpha \cap V = \emptyset$ for all but finitely many α.

We denote the set of divisors on X by $\mathbf{Div}(\boldsymbol{X})$*, which is a group under addition.*

Example 3.6. *If* $\dim(X) = 1$ *(i.e. X is a Riemann surface), then a divisor is just a collection of points with some multiplicities (i.e. the multiplicities being the coefficients a_α).*

Definition 3.24. *We say that a divisor D is **effective** if $a_\alpha \geq 0$ for all α.*

Definition 3.25. *If $f \in H^0(X, K^*)$, we define the **divisor associated to f** via:*
$$(f) := \sum_{Y} \mathrm{ord}_Y(f) Y$$
where the sum is over all $Y \subset X$ irreducible analytic hypersurfaces (recall here that K^* is the sheaf of meromorphic functions which are not identically zero).

To check that this (f) is actually a divisor we need to check that the sum is locally finite. But this is the case, as given $x \in X$, then locally about x we have $f = g/h$, and there are only finitely many Y with $\mathrm{ord}_Y(g) \neq 0$ (seen by writing g as a product of irreducibles).

Note: (f) is effective $\iff$ f is holomorphic.

Definition 3.26. *We call a divisor D a **principle divisor** if $D = (f)$ for some $f \in H^0(X, K^*)$.*

*We say that divisors D, D' are **linearly equivalent**, and write $D \sim D'$, if $D - D'$ is a principle divisor.*

Note: The relation $\sim$ of linear equivalence is transitive because $(f) + (g) = (fg)$, which comes from $\mathrm{ord}_Y(fg) = \mathrm{ord}_Y(f) + \mathrm{ord}_Y(g)$.

There is an natural inclusion of sheaves $\mathscr{O}^* \hookrightarrow K^*$ as every holomorphic function is meromorphic. Thus we obtain $\frac{K^*}{\mathscr{O}^*}$, the quotient sheaf, obtained by sheafifying the presheaf defined by $U \mapsto \frac{K^*(U)}{\mathscr{O}^*(U)}$ (we need to sheafify since the quotient only forms a presheaf in general). A global section $f \in H^0\left(X, \frac{K^*}{\mathscr{O}^*}\right)$ thus consists of an open cover $\{U_\alpha\}_\alpha$ of X and meromorphic functions $f_\alpha \in K^*(U_\alpha)$ with $\left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha \cap U_\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$, whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Proposition 3.5. *There is an isomorphism* $H^0\left(X, \frac{K^*}{\mathcal{O}^*}\right) \cong \mathrm{Div}(X)$.

Proof. Let $f \in H^0\left(X, \frac{K^*}{\mathcal{O}^*}\right)$. Then we know f is given by meromorphic functions $(f_\alpha)_\alpha$ on an open cover $(U_\alpha)_\alpha$ of X as detailed above. Now if Y is an irreducible analytic hypersurface with $Y \cap (U_\alpha \cap U_\beta) \neq \emptyset$, we have

$$\mathrm{ord}_Y(f_\alpha) = \mathrm{ord}_Y(f_\beta)$$

since $\left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha \cap U_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ is holomorphic and so $\mathrm{ord}_Y\left(\frac{f_\alpha}{f_\beta}\right) = 0$ (since if we take the order for a point $p \in Y \cap (U_\alpha \cap U_\beta)$ this will be 0 as f_α/f_β is holomorphic in $U_\alpha \cap U_\beta$).

Thus we may define $\mathrm{ord}_Y(f) := \mathrm{ord}_Y(f_\alpha)$ for any U_α with $Y \cap U_\alpha \neq \emptyset$, and thus this gives a map $H^0\left(X, \frac{K^*}{\mathcal{O}^*}\right) \to \mathrm{Div}(X)$ via

$$f \longmapsto \sum_Y \mathrm{ord}_Y(f) Y.$$

This is clearly a group homomorphism, by the additivity of ord.

We next construct an inverse to the above map. Suppose $D = \sum_\alpha a_\alpha Y_\alpha$ is a divisor on X. Consider Y_α. Then there is an open cover $\{U_\beta\}_\beta$ of X and $g_{\alpha\beta} \in \mathcal{O}(U_\beta)$ such that

$$Y_\alpha \cap U_\beta = g_{\alpha\beta}^{-1}(0)$$

since Y is an irreducible analytic hypersurface (with say $g_{\alpha\beta} = 1$ in $Y_\alpha \cap U_\beta = \emptyset$). Now set

$$f_\beta := \prod_\alpha g_{\alpha\beta}^{a_\alpha}$$

which is a finite product as D is locally finite. Now since $g_{\alpha\beta}$ and $g_{\alpha\gamma}$ define the same hypersurface in $U_\beta \cap U_\gamma$, we have

$$\frac{g_{\alpha\beta}}{g_{\alpha\gamma}} \in \mathcal{O}^*(U_\beta \cap U_\gamma)$$

by the weak-nullstellensatz (Theorem 3.2). Thus the f_β glue to give a section of $H^0\left(X, \frac{K^*}{\mathcal{O}^*}\right)$, and so this construction gives a map $\mathrm{Div}(X) \to H^0\left(X, \frac{K^*}{\mathcal{O}^*}\right)$. These two maps are then clearly inverses of one another [**Exercise** to check details] and so we are done.

□

Remark: We shall say that $D \in \mathrm{Div}(X)$ is given by **local data** (U_α, f_α), using the above construction in Proposition 3.5.

Theorem 3.5. $\exists$ *a group homomorphism* $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ *via* $D \mapsto \mathcal{O}(D)$*, the kernel of which is precisely the set of principle divisors.*

Proof. Let $D \in \mathrm{Div}(X)$ be given by local data (U_α, f_α) as per Proposition 3.5. Let $\varphi_{\alpha\beta} := \left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha \cap U_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. These then satisfy the cocycle conditions ($\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = \mathrm{id}$) and so generate/give an element of Pic(X) $\cong H^1(X, \mathcal{O}^*)$. We first check thar this is well-defined, i.e. independent of the choice of local data defining D.

So suppose $(U_\alpha, \tilde{f}_\alpha)$ is alternative local data (same U_α by the construction in Proposition 3.5). Then we have $f_\alpha = s_\alpha \tilde{f}_\alpha$ for some $s_\alpha \in \mathcal{O}^*(U_\alpha)$. The new transition functions defining an element of Pic(X) are:

$$\tilde{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta} \cdot \frac{s_\beta}{s_\alpha}.$$

Then $(U_\alpha, s_\beta/s_\alpha)$ satisfy the cocycle conditions, thus giving a line bundle L with a nowhere vanishing section s induced by the s_α. Then if the line bundles defined by $(U_\alpha, \varphi_{\alpha\beta})$ and $(U_\alpha, \tilde{\varphi}_{\alpha\beta})$ are, say, H and $\tilde{H}$, then

$$\tilde{H} \cong H \otimes L$$

as $\tilde{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta} \cdot \frac{s_\beta}{s_\alpha}$, and we know that the transition functions associated to a tensor product are just the products of the transition functions for the corresponding (line) bundles.

But L has a nowhere vanishing section, and hence L must be (isomorphic to) the trivial line bundle. Hence we have

$$\tilde{H} \cong H =: \mathcal{O}(D)$$

and so this map is well-defined.

Next we need to check that the map is a group homomorphism. So let $D, \tilde{D} \in \mathrm{Div}(X)$ be given by local data (U_α, f_α) and $(U_\alpha, \tilde{f}_\alpha)$. Then $D + \tilde{D}$ is the divisor with local data given by $(U_\alpha, f_\alpha \tilde{f}_\alpha)$ (since Proposition 3.5 gives a homomorphism) and so

$$\mathcal{O}(D + \tilde{D}) \cong \mathcal{O}(D) \otimes \mathcal{O}(\tilde{D})$$

for the same reason (the transition functions of a tensor product is exactly the product of the transition functions). So $\mathcal{O}$ is a homomorphism.

Finally, we need to show that the kernel of this map is the set of principle divisors. For one inclusion, suppose $D = (f)$, $f \in H^0(X, K^*)$, is a principle divisor. Then we can take (U_α, f_α) to be the local data (f_α being the meromorphic functions determining f as usual). Then

$$\varphi_{\alpha\beta} = \left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha \cap U_\beta} = \mathrm{id}$$

in $H^0\left(X, \frac{K^*}{\mathcal{O}^*}\right)$ since this function is holomorphic. Thus $\mathcal{O}(D)$ has trivial transition functions and hence $\mathcal{O}(D) \cong \mathcal{O}$ (the sheaf of holomorphic functions, which is the identity in Pic(X)).

For the reverse inclusion, suppose $\mathcal{O}(D) \cong \mathcal{O}$ (so D is in the kernel). So $\exists s$ a global nowhere vanishing holomorphic section of $\mathcal{O}(D)$. Suppose $\mathcal{O}(D)$ has transition functions $(U_\alpha, \varphi_{\alpha\beta})$, and so D is given by (U_α, f_α) where $\varphi_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ (locally on $U_\alpha \cap U_\beta$). Set $s_\alpha := s|_{U_\alpha}$, and so $s_\alpha = \varphi_{\alpha\beta} s_\beta$ [see Example Sheet 3 for more elaboration]. Then

$$\frac{s_\alpha}{s_\beta} = \varphi_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

Thus g defined by $g|_{U_\alpha} := \frac{f_\alpha}{s_\alpha}$ is a well-defined global meromorphic function on X, since the above tells us that on $U_\alpha \cap U_\beta$ we have $\frac{f_\alpha}{s_\alpha} = \frac{f_\beta}{s_\beta}$. Then $D = (g)$, since the s_α are nowhere vanishing.

Thus the reverse inclusion has been proven, and so $\ker(\text{Div}(X) \to \text{Pic}(X)) = \{\text{principle divisors of } X\}$. $\square$

Exercise: Show that there is an exact sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow K^* \longrightarrow \frac{K^*}{\mathcal{O}^*} \longrightarrow 0$$

and use the associated long exact sequence in cohomology to give another proof of Theorem 3.5.

So we have a map between these two groups, and so you might wonder if we can go the other way. In some cases we can:

Proposition 3.6. *For any $s \in H^0(X, L)\backslash\{0\}$, $\exists$ an associated $Z(s) \in \text{Div}(X)$.*

Proof. Fix a trivialisation for L, $\pi : L \to X$. Then:

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{C}$$

with cocycle data $(U_\alpha, \varphi_{\alpha\beta})$. Set $f_\alpha := \varphi_\alpha(s|_{U_\alpha}) \in \mathcal{O}(U_\alpha)$, which is not identically 0. Then we have

$$f_\alpha f_\beta^{-1} = \varphi_\alpha(s|_{U_\alpha})\varphi_\beta(s|_{U_\beta})^{-1} = \varphi_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

Thus one obtains $Z(s) \in \text{Div}(X)$ determined by the local data (U_α, f_α). In addition we also see [**Exercise** to check]

$$Z(s_1 + s_2) = Z(s_1) + Z(s_2).$$

$\square$

Proposition 3.7. *We have the following:*

(i) Let $s \in H^0(X, L)\backslash\{0\}$. Then $\mathcal{O}(Z(s)) \cong L$.

(ii) If D is effective, $\exists\, s \in H^0(X, \mathcal{O}(D))\backslash\{0\}$ with $Z(s) = D$.

Proof. (i): Let L have trivialisation $(U_\alpha, \varphi_\alpha)$. Then $Z(s)$ is given by $f \in H^0\left(X, \frac{K^*}{\mathcal{O}^*}\right)$, where $f_\alpha = f|_{U_\alpha} = \varphi_\alpha(s|_{U_\alpha})$. Then $\mathcal{O}(Z(s))$ is the line bundle with associated cocycle data being $(U_\alpha, f_\alpha f_\beta^{-1})$. But:

$$f_\alpha f_\beta^{-1} = \varphi_\alpha(s|_{U_\alpha})\varphi_\beta(s|_{U_\beta})^{-1} = \varphi_{\alpha\beta}$$

as in Proposition 3.6, and thus this line bundle is just L as the cocycle data is the same.

(ii): Let $D \in \text{Div}(X)$ be given by (U_α, f_α), with $f_\alpha \in K^*(U_\alpha)$. Then as D is effective we know the f_α are holomorphic. The line bundle $\mathcal{O}(D)$ is associated to the cocycle data $(U_\alpha, \varphi_{\alpha\beta})$, where $\varphi_{\alpha\beta} = \left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha \cap U_\beta}$. Then $f_\alpha \in \mathcal{O}(U_\alpha)$ glue to a global section $s \in H^0(X, \mathcal{O}(D))$ as $f_\alpha = \varphi_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta$. Moreover,

$$Z(s)|_{U_\alpha} = Z(s|_{U_\alpha}) = Z(f_\alpha) = D \cap U_\alpha$$

and so $Z(s) = D$ (note by $D \cap U_\alpha$ we just mean $\sum_\beta a_\beta \left(Y_\beta \cap U_\alpha\right)$ if $D = \sum_\beta a_\beta Y_\beta$). $\square$

Note that the s found in Proposition 3.7(ii) is not unique: if $\lambda \in H^0(X, \mathcal{O}^*)$ (e.g. $\lambda \in C^*$) then $Z(\lambda s) = Z(s)$. It turns out on non-compact manifolds s is highly non-unique, but on compact ones we get uniqueness up to multiplication by such a λ.

Corollary 3.1. *Let $s \in H^0(X, L)$ and $\tilde{s} \in H^0(X, \tilde{L})$. Then:*

$$Z(s) \sim Z(\tilde{s}) \quad \Longleftrightarrow \quad L = \tilde{L}.$$

Proof. This follows as $\mathcal{O}(Z(s)) \cong L$ and we know $\ker(\mathcal{O}) = \{\text{principle divisors}\}$. Thus $\mathcal{O}(D) \cong \mathcal{O} \iff D$ is principle.

□

Recall the exponential s.e.s:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

where $\underline{\mathbb{Z}}$ is denotes the constant sheaf. The l.e.s in cohomology and the fact that $\text{Pic}(X) \cong H^1(X, \mathcal{O}^*)$ (from Proposition 3.3) gives a map (via the first chain map):

$$c_1 : \text{Pic}(X) \to H^2(X, \mathbb{Z}).$$

Definition 3.27. *For $L \in \text{Pic}(X)$, we call $c_1(L) \in H^2(X, \mathbb{Z})$ the **first Chern class of L**.*

We will return to Chern classes later in the course.

4. Kähler Manifolds

Recall: A complex manifold X is **projective** if it is biholomorphic to a closed submanifold of $\mathbb{P}^m$ for some m.

Definition 4.1. *We say that a line bundle on X is **ample** if there is an embedding i of X into $\mathbb{P}^m$ for some m and $\exists$ a $k \in \mathbb{Z}_{>0}$ such that*
$$L^{\otimes k} \cong i^*(\mathcal{O}(1))$$
where $\mathcal{O}(1)$ is the hyperplane line bundle on $\mathbb{P}^m$.

Kähler geometry (in part) gives a differential geometric interpretation of amplitude (i.e. ampleness), which we shall look into more now.

4.1. Kähler Linear Algebra.

Just as we did for complex structures we will start with some linear algebra. The goal is to put Riemannian metrics on complex manifolds which interact well with the complex structure.

Let V be a real finite dimensional vector space and let $J : V \to V$ be a complex structure (so $J^2 = -\text{id}$). Let $\langle \cdot, \cdot \rangle$ be an inner product on V.

Definition 4.2. *We say that $\langle \cdot, \cdot \rangle$ is **compatible with the complex structure** J if:*
$$\langle J(u), J(v) \rangle = \langle u, v \rangle \qquad \forall u, v \in V.$$

Definition 4.3. *If $\langle \cdot, \cdot \rangle$ is compatible with J, we then define the **fundamental (2-)form** ω by:*
$$\omega(u, v) := \langle J(u), v \rangle.$$

Note that ω is antisymmetric, since:
$$\omega(u, v) = \langle J(u), v \rangle = \langle J^2(u), J(v) \rangle = \langle -u, J(v) \rangle = -\langle J(v), u \rangle = -\omega(v, u)$$
where we have used the compatibility assumption and the symmetry of the inner product.

We now extend these notions to the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The inner product extends to a hermitian inner product via:
$$\langle \lambda u, \mu v \rangle_{\mathbb{C}} := \lambda \overline{\mu} \langle u, v \rangle \qquad \forall \lambda, \mu \in \mathbb{C},\ u, v \in V$$
and using that any $\alpha \in V_{\mathbb{C}}$ can be written as $\alpha = \alpha_1 + i\alpha_2$ for $\alpha_1, \alpha_2 \in V$.

We assume that $\langle \cdot, \cdot \rangle$ is compatible with J throughout, so that ω exists. We can then extend ω to $V_{\mathbb{C}}$ via $\omega(u, v) := \langle J(u), v \rangle_{\mathbb{C}}$, where J here is the natural extension of J onto $V_{\mathbb{C}}$ via acting on the V part of $V_{\mathbb{C}}$.

Lemma 4.1. *We have the following:*

(i) The decomposition $V_{\mathbb{C}} := V^{(1,0)} \oplus V^{(0,1)}$ from before is orthogonal w.r.t. the hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$.

(ii) $\omega \in \Lambda^{1,1} V_{\mathbb{C}}^*$, i.e. the fundamental form is a $(1,1)$-form.

Proof. (i): Let $u \in V^{(1,0)}$, $v \in V^{(0,1)}$. Then $J(u) = iu$ and $J(v) = -iv$, and so

$$\langle u, v \rangle_{\mathbb{C}} = \langle J(u), J(v) \rangle_{\mathbb{C}} = \langle iu, -iv \rangle_{\mathbb{C}} = i^2 \langle u, v \rangle_{\mathbb{C}} = -\langle u, v \rangle_{\mathbb{C}} \quad \Longrightarrow \quad \langle u, v \rangle_{\mathbb{C}} = 0$$

where we have used compatibility and the fact that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is conjugate-linear in the second component.

(ii): Let $u, v \in V^{(1,0)}$. Then,

$$\omega(u, v) = \langle J(u), v \rangle_{\mathbb{C}} = \langle J^2(u), J(v) \rangle_{\mathbb{C}} = \omega(J(u), J(v)\rangle$$

by compatibility. So hence

$$\omega(u, v) = \omega(J(u), J(v)\rangle = \omega(iu, iv) = i^2 \omega(u, v) = -\omega(u, v) \quad \Longrightarrow \quad \omega(u, v) = 0.$$

Similarly if $u, v \in V^{(0,1)}$ then we have $\omega(u, v) = 0$. So hence ω is only non-zero on $V^{(1,0)} \times V^{(0,1)}$ and $V^{(0,1)} \times V^{(1,0)}$, and thus $\omega \in \Lambda^{1,1} V_{\mathbb{C}}^*$.

□

4.2. **Kähler Geometry.**

Now let X be a complex manifold with almost complex structure (a.c.s) J. Recall from differential geometry:

Definition 4.4. *A **Riemannian metric** g on X is a section of $T^*X \otimes T^*X$ such that for all $x \in X$, $g_x : T_xX \times T_xX \to \mathbb{R}$ is an inner product.*

Definition 4.5. *A Riemannian metric g on X is **compatible with J** if for all $x \in X$, the inner product g_x on T_xX is compatible with the complex structure $J_x : T_xX \to T_xX$.*

We can then define ω, the **fundamental form**, by

$$\omega(u, v) := g(J(u), v).$$

ω extends $\mathbb{C}$—linearly to a $(1,1)$—form $\omega \in \Lambda^{1,1}(T^*X)_{\mathbb{C}}$ as we saw in the linear algebra case.

The extension $g_{\mathbb{C}}$ of g gives a hermitian metric (by definition) on $(TX)_{\mathbb{C}}$ and hence on $TX^{(1,0)}$. Suppose on X we have holomorphic coordinates $z_1, \ldots, z_n$. Then $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ form a local holomorphic frame for $T^*X^{(1,0)}$. Let:

$$h_{jk} := 2g_{\mathbb{C}}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right).$$

Exercise: Show that $(h_{jk})_{jk}$ is a hermitian matrix and that

$$\omega = \frac{i}{2}\sum_{j,k} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k.$$

Definition 4.6. *We say that ω is a **Kähler form** (or **Kähler metric**) on X if in addition we have $\mathrm{d}\omega = 0$, i.e. ω is closed.*

*We then say $[\omega] \in H^2(X,\mathbb{R})$ is a **Kähler class**.*

Definition 4.7. *We call a complex manifold a **Kähler manifold** if it has a Kähler metric. We write (X,ω) for the Kähler manifold.*

Example 4.1. *On $\mathbb{C}^n$ with coordinates $z_1,\dots,z_n$, $\omega = \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ is a Kähler metric.*

Example 4.2. *By a standard partition of unity argument, any complex manifold admits a hermitian metric. Moreover if g is a Riemannian metric, then*

$$\tilde{g}(u,v) = g(u,v) + g(J(u),J(v))$$

is a hermitian metric compatible with J. Then if $\dim(X) = 1$ (i.e. a Riemann surface) then every $(1,1)$–form is automatically closed, and thus we get lots of Kähler forms on Riemann surfaces.

Note: Knowing any two of g, J, ω determines the third completely.

Remark: Any Kähler metric induces a symplectic form on X. Thus Kähler geometry (i.e. the study of complex manifolds with Kähler forms) lies in the intersection of complex geometry, Riemannian geometry (as have a Riemannian metric) and symplectic geometry. This is in some sense why the theory of Kähler geometry is so rich.

Crucially, (complex) projective space has a Kähler form, as per the following Example 4.3.

Example 4.3 (The Fubini-Study metric on $\mathbb{P}^n$). *Let $U \subset \mathbb{P}^n$ be open and let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ be the natural projection. Suppose $s : U \to \mathbb{C}^{n+1}$ is a holomorphic lift of π, i.e. $\pi(s(z)) = z$ for all $z \in U$ (e.g. if $U = U_j = \{[z_0 : \cdots : z_n \mid z_j \neq 0\}$, then $s([z_0 : \cdots : z_n]) = \frac{1}{z_j}(z_0,\dots,z_n)$).*

Let

$$\omega_{FS}|_U := \frac{i}{2\pi}\partial\overline{\partial}\left[\log(\|s\|^2)\right]$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{C}^{n+1}$. We need to check that this is well-defined (i.e. agrees on overlaps), closed and positive definite.

TO see if is well-defined, choose another s' defined on U'. Then we know $s' = fs$ for some $f \in \mathscr{O}^*(U \cap U')$, and

$$\frac{i}{2\pi}\partial\overline{\partial}\left[\log(\|s'\|^2)\right] = \frac{i}{2\pi}\partial\overline{\partial}\left[\log(|f|^2\|s\|^2)\right] = \frac{i}{2\pi}\partial\overline{\partial}\left[\log(|f|^2) + \log(\|s\|^2)\right] = \omega_{FS}|_U$$

as $\partial\overline{\partial}\left[\log(|f|^2)\right] = \partial\overline{\partial}\left[\log(f) + \log(\overline{f})\right] = 0$. Thus ω_{FS} is well-defined on all of $\mathbb{P}^n$.

Next, to see that it is closed, note that $(\partial+\overline{\partial})(\overline{\partial}-\partial) = \partial^2 + \overline{\partial}^2 + \partial\overline{\partial} - \overline{\partial}\partial = 2\partial\overline{\partial}$ since $\partial^2 = 0 = \overline{\partial}^2$ and $\partial\overline{\partial} = -\overline{\partial}\partial$ and so

$$2\omega_{FS} = \frac{i}{2\pi}(\partial+\overline{\partial})(\overline{\partial}-\partial)\left[\log(\|s\|^2)\right] = \mathrm{d}\left(\frac{i}{2\pi}(\overline{\partial}-\partial)\log(\|s\|^2)\right)$$

and so ω_{FS} is exact, and thus closed.

Finally we know that locally we can write

$$\omega_{FS} = \frac{i}{2}\sum_{j,k} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$$

and to see ω_{FS} is positive definite we need to show that $(h_{jk})_{jk}$ is a positive definite hermitian matrix. To see this, we work on U_0 (the proof for the other U_j is identical). Set $w_j = \frac{z_j}{z_0}$. Then:

$$\begin{aligned}
\omega_{FS}|_{U_0} &= \frac{i}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_j |w_j|^2\right) \\
&= \frac{i}{2\pi}\partial\left(\frac{\sum_j w_j \mathrm{d}\overline{w}_j}{1+\sum_j |w_j|^2}\right) \\
&= \frac{i}{2\pi}\left(\frac{\sum_j \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j}{1+\sum_j |w_j|^2} - \frac{\left(\sum_j \overline{w}_j \mathrm{d}w_j\right)\wedge\left(\sum_j w_j \mathrm{d}\overline{w}_j\right)}{\left(1+\sum_j |w_j|^2\right)^2}\right) \\
&= \frac{i}{2\pi}\sum_{j,k}\underbrace{\frac{\left(1+\sum_l |w_l|^2\right)\delta_{jk} - \overline{w}_j w_k}{1+\sum_l |w_l|^2}}_{=:h_{jk}}\mathrm{d}w_j \wedge \mathrm{d}\overline{w}_k \\
&= \frac{i}{2\pi}\sum_{j,k} h_{jk}\mathrm{d}w_j \wedge \mathrm{d}\overline{w}_k.
\end{aligned}$$

Then if $0 \neq u \in \mathbb{C}^n$ (ignoring the positive denominator of the h_{jk}):

$$\begin{aligned}
u^T(h_{jk})_{jk}\overline{u} &= \langle u,u\rangle + \langle w,w\rangle\langle u,u\rangle - \langle u,w\rangle\langle w,u\rangle \\
&= \langle u,u\rangle + \langle w,w\rangle\langle u,u\rangle - |\langle w,u\rangle|^2 \\
&> 0 \quad \text{by Cauchy-Schwarz.}
\end{aligned}$$

Hence this is positive definite and so ω_{FS} is a Kähler metric on $\mathbb{P}^n$. □

Proposition 4.1. *Let (X,ω) be a Kähler manifold. Then any complex submanifold $i : Y \hookrightarrow X$ is Kähler.*

Proof. Since d commutes with pullbacks we have

$$\mathrm{d}(i^*\omega) = i^*(\mathrm{d}\omega) = i^*(0) = 0.$$

Moreover clearly $i^*\omega$ is positive definite on Y since ω is on X. Thus $i^*\omega$ is a Kähler form on Y and so Y is Kähler.

□

Corollary 4.1. *Any projective manifold is Kähler.*

Proof. Apply Proposition 4.1 since $\mathbb{P}^n$ is Kähler for any n. □

In general, using the hermitian metric $h = g_{\mathbb{C}}$ on $TX^{(1,0)}$, choose a unitary frame $\{\varphi_1,\dots,\varphi_n\}$ of $T^*X^{(1,0)}$ on a neighbourhood U of $x \in X$ so that $h = \sum_j \varphi_j \otimes \overline{\varphi}_j$. Let $\eta_j = \mathrm{Re}(\varphi_j)$, $\xi_j = \mathrm{Im}(\varphi_j)$. One can then check:

$$g = \mathrm{Re}\left(\sum_j (\eta_j + i\xi_j)\otimes(\eta_j - i\xi_j)\right) = \sum_j (\eta_j\otimes\eta_j + \xi_j\otimes\xi_j)$$

with volume form

$$\mathrm{dVol} = \eta_1\wedge\xi_1\wedge\cdots\wedge\eta_n\wedge\xi_n.$$

Now,

$$\omega = \frac{i}{2\pi}\sum_j (\eta_j + i\xi_j)\wedge(\eta_j + i\xi_j) = \frac{i}{2\pi}\sum_j \eta_j\wedge\xi_j.$$

Thus we see:

$$\frac{\omega^n}{n!} = \mathrm{dVol}$$

(up to a factor of 2π), and so a Kähler form gives a volume form. In particular, when X is compact we have

$$\int_X \omega^n > 0.$$

So if ω is a Kähler metric associated to a Riemannian metric g, we have a corresponding volume form given by $\mathrm{dVol} = \omega^n/n!$.

Proposition 4.2. *If X is a compact Kähler manifold, then $\dim\left(H^{2q}_{\mathrm{dR}}(X,\mathbb{R})\right) > 0$ for all $q \in \{1,\dots,\frac{1}{2}\dim_{\mathbb{R}}(X)\}$, i.e. all the even Betti numbers are > 0.*

Proof. Let ω be a Kähler metric, and set $\tau = \overbrace{\omega \wedge \cdots \wedge \omega}^{q \text{ times}}$. Then $\mathrm{d}\tau = 0$ as $\mathrm{d}\omega = 0$, and so $[\tau] = H^{2q}_{\mathrm{dR}}(X,\mathbb{R})$. We just need to show that this class is non-zero, i.e. τ is not exact.

Suppose $\tau = \mathrm{d}\sigma$ for some $\sigma \in \mathscr{A}_{\mathbb{R}}(X)$. Then we would have

$$\int_X \omega^n = \int_X \omega^{n-q} \wedge \tau = \int_X \mathrm{d}(\omega^{n-q} \wedge \sigma) = 0$$

and this is 0 by Stoke's theorem (as X is compact so has no boundary). But we know $\int_X \omega^n > 0$, and so this is a contradiction. Hence τ is not exact and so $[\tau] \in H^{2q}_{\mathrm{dR}}(X,\mathbb{R})\backslash\{0\}$.

□

Thus Proposition 4.2 tells us that there is a topological obstruction for a compact complex manifold to be Kähler, namely all even order de Rham cohomology groups must be non-trivial - see Example Sheet 3.

Remark: We saw in Corollary 4.1 that every (smooth) projective manifold is Kähler. Recall to any $L \in \mathrm{Pic}(X)$ we defined $c_1(L) \in H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{R})$. We are working towards the *Kodaira embedding theorem*, which states that on a compact complex manifold, a class $\alpha \in H^2(X,\mathbb{Z})$ is a Kähler class (i.e. $\exists\omega$ Kähler with $\omega \in \alpha$) if and only if $\alpha = c_1(L)$ for some ample $L \in \mathrm{Pic}(X)$. This gives a complex differential geometric interpretation of ampleness, and characterises which compact Kähler manifolds are projective. [These results also won the Fields Medal at one point.]

Proposition 4.3. *Let ω be a $(1,1)$-form associated to a hermitian metric h on X. Then:*

$$\mathrm{d}\omega = 0 \quad \Longleftrightarrow \quad \exists \text{ holomorphic coordinates } z_1,\dots,z_n \text{ about } x \text{ s.t. locally}$$

$$\omega = \frac{i}{2}\sum_{j,k} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k \text{ with } h_{jk} = \delta_{jk} + O(|z|^2)$$

where δ_{jk} is the Dirac-delta. Thus:

$$\omega \text{ is Kähler} \quad \Longleftrightarrow \quad \omega = \omega_0 + O(|z|^2)$$

where ω_0 is the usual Kähler form on $\mathbb{C}^n$.

Remark: This result is very useful and we will use it a fair bit - it tells us that any identity valid on $\mathbb{C}^n$ (with its usual Kähler metric) which only involves the metric h and its first order derivatives, is true on any Kähler manifold (essentially because the above tells us we can kill off the linear term).

Proof. We will use summation convention, but i will also be $\sqrt{-1}$, not an index.

$(\Leftarrow)$: Say $\omega = \frac{i}{2}h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$. Then

$$\mathrm{d}\omega = \frac{i}{2}\frac{\partial h_{jk}}{\partial z_l}\mathrm{d}z_l \wedge \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k + \frac{i}{2}\frac{\partial h_{jk}}{\partial \overline{z}_l}\mathrm{d}\overline{z}_l \wedge \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k.$$

Thus if $h_{jk} = \delta_{jk} + O(|z|^2)$, then $\frac{\partial h_{jk}}{\partial z_l}(x) = 0$ and so $\mathrm{d}\omega = 0$.

<u>$(\Rightarrow)$</u> : Suppose $\mathrm{d}\omega = 0$. Write $\omega = \frac{i}{2} h_{jk} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$. By a linear change of coordinates we may assume $h_{jk}(x) = \delta_{jk}$ (at this <u>one</u> point x). The Taylor series expansion of h_{jk} then looks like

$$h_{jk} = \delta_{jk} + a_{jkl} z_l + b_{jkl} \overline{z}_l + O(|z|^2).$$

As h is hermitian we have $h_{jk} = \overline{h}_{kj}$ and thus $b_{jkl} = \overline{a}_{kjl}$. But then as $\mathrm{d}\omega = 0$,

$$0 = a_{jkl} \mathrm{d}z_l \wedge \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k + b_{jkl} \mathrm{d}\overline{z}_l \wedge \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$$

and thus we must have (due to these wedge products forming a basis) $a_{jkl} = a_{lkj}$ and $b_{jkl} = b_{jlk}$.

Now let $\zeta_k = z_k + \frac{1}{2} a_{jkl} z_j z_l$, which is a valid change of coordinates in a neighbourhood of x. Then:

$$\mathrm{d}\zeta_k = \mathrm{d}z_k + \frac{1}{2} a_{jkl} \left(z_j \mathrm{d}z_l + z_l \mathrm{d}z_j \right) = \mathrm{d}z_k + a_{jkl} z_j \mathrm{d}z_l$$

$$\mathrm{d}\overline{\zeta}_k = \mathrm{d}\overline{z}_k + \frac{1}{2} \overline{a}_{jkl} \left(\overline{z}_j \mathrm{d}\overline{z}_l + \overline{z}_l \mathrm{d}\overline{z}_j \right) = \mathrm{d}\overline{z}_k + \overline{a}_{jkl} \overline{z}_j \mathrm{d}\overline{z}_l$$

using the symmetries of a_{jkl}, and so now we can compute

$$\begin{aligned} \mathrm{d}\zeta_k \wedge \mathrm{d}\overline{\zeta}_k &= \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k + \overline{a}_{jkl} \overline{z}_j \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_l + a_{jkl} z_j \mathrm{d}z_l \wedge \mathrm{d}\overline{z}_k + O(|z|^2) \\ &= \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k + a_{jkl} z_l \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k + b_{jkl} \overline{z}_l \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k + O(|z|^2) \\ &= \frac{2}{i} \omega + O(|z|^2) \end{aligned}$$

and so as in these coordinates $\mathrm{d}\zeta_k \wedge \mathrm{d}\overline{\zeta}_k = \omega_0$, we are done.

□

4.3. **Kähler Identities.**

Let (X, g) be an oriented Riemannian manifold of dimension $2n$. The exterior derivative $\mathrm{d} : \mathscr{A}^k \to \mathscr{A}^{k+1}$ we know satisfies $\mathrm{d}^2 = 0$. Let dVol be the volume form associated to g.

Definition 4.8. *The* ***Hodge star operator*** *is the map* $* : \mathscr{A}^k \to \mathscr{A}^{2n-k}$ *which is uniquely determined by the relation:*

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle_g \ \mathrm{dVol} \qquad \forall \alpha, \beta \in \mathscr{A}^k.$$

Definition 4.9. *The* ***L^2-adjoint of*** **d** *is the map* $\mathrm{d}^* : \mathscr{A}^k \to \mathscr{A}^{k-1}$ *defined by:*

$$\mathrm{d}^* := - * \mathrm{d} * .$$

We will see later that d^* is actually the adjoint of d with respect to an appropriate L^2 inner product.

Definition 4.10. *The* ***Laplacian*** $\Delta_{\mathrm{d}} : \mathscr{A}^k \to \mathscr{A}^k$ *is defined by:*

$$\Delta_{\mathrm{d}} := \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*.$$

Now suppose X is a complex manifold of dimension n, with Riemannian metric g compatible with J. Then the Hodge star operator extends naturally to $* : \mathscr{A}^k_{\mathbb{C}} \to \mathscr{A}^{2n-k}_{\mathbb{C}}$ in such a way so that

$$\alpha \wedge (*\beta) = g_{\mathbb{C}}(\alpha, \beta) \cdot \mathrm{dVol}.$$

Remark: If $\alpha \in \mathscr{A}^k_{\mathbb{C}}$ we have $**\alpha = (-1)^{k(2n-k)}\alpha$, and thus $*^{-1} = (-1)^{k(2n-k)}*$.

Now write $\mathrm{d} = \partial + \overline{\partial}$ as usual, where $\partial : \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q}$ and $\overline{\partial} : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}$.

Definition 4.11. *Define the* ***L^2-adjoints of ∂*** *and* ***$\overline{\partial}$*** *by:*

$$\partial^* := -*\partial* \qquad \text{and} \qquad \overline{\partial}^* = -*\overline{\partial}*$$

and define the associated ***Laplacian****'s:*

$$\Delta_{\partial} := \partial^*\partial + \partial\partial^* \qquad \text{and} \qquad \Delta_{\overline{\partial}} := \overline{\partial}^*\overline{\partial} + \overline{\partial}\,\overline{\partial}^*.$$

Definition 4.12. *If ω is Kähler, define the* ***Lefschetz operator*** $L : \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q+1}$ *by:*

$$L(\alpha) := \alpha \wedge \omega.$$

Also define the ***contraction*** *(or* ***inverse Lefschetz operator****)* $\Lambda : \mathscr{A}^{p,q} \to \mathscr{A}^{p-1,q-1}$ *by*

$$\Lambda := *^{-1}L*.$$

Definition 4.13. *For $\alpha, \beta \in \mathscr{A}^{p,q}$ we define the* ***L^2-inner product*** *by:*

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \alpha \wedge (*\beta) \quad \left(\equiv \int_X g_{\mathbb{C}}(\alpha, \beta)\, \mathrm{dVol}\right).$$

Lemma 4.2. *Let X be a compact Kähler manifold. Then the operators ∂^* and $\overline{\partial}^*$ are the L^2-adjoints of the operators ∂ and $\overline{\partial}$ respectively, i.e.*

- *if $\alpha \in \mathscr{A}^{p,q}$ and $\beta \in \mathscr{A}^{p+1,q}$ then*
$$\langle \partial\alpha, \beta \rangle_{L^2} = \langle \alpha, \overline{\partial}\beta \rangle_{L^2}$$
- *if $\alpha \in \mathscr{A}^{p,q}$ and $\beta \in \mathscr{A}^{p,q+1}$ then*
$$\langle \overline{\partial}\alpha, \beta \rangle_{L^2} = \langle \alpha, \overline{\partial}\beta \rangle_{L^2}.$$

Proof. We only prove the first identity as the second is very similar. By Stoke's theorem we have (as X is compact)

$$0 = \int_X \mathrm{d}(\alpha \wedge (*\beta)) = \int_X \partial(\alpha \wedge (*\beta))$$

where the second equality is because $\alpha \wedge (\beta) \in \mathscr{A}^{p+(n-(p+1)),n} = \mathscr{A}^{n-1,n}$ and so $\overline{\partial}(\alpha \wedge (*\beta)) = 0$. So,

$$0 = \int_X \partial(\alpha \wedge (*\beta)) = \int_X \partial\alpha \wedge (*\beta) + (-1)^k \alpha \wedge \partial(*\beta)$$

where $p+q=k$. Thus

$$\begin{aligned}
\langle \partial\alpha, \beta\rangle_{L^2} &= \int_X \partial\alpha \wedge (*\beta) \\
&= (-1)^{k+1}\int_X \alpha \wedge (\partial * \beta) \\
&= (-1)^{k+1}\int_X \alpha \left[(-1)^{k(2n-k)} * \underbrace{(*\partial*\beta)}_{=-\partial^*\beta}\right] \qquad \text{as } (-1)^{k(2n-k)} ** = \mathrm{id} \\
&= \underbrace{-(-1)^{k+1+k(2n-k)}}_{=1\ \forall k}\int_X \alpha \wedge *(\partial^*\beta) \\
&= \int_X \alpha \wedge (*\partial^*\beta) \\
&=: \langle \alpha, \partial^*\beta\rangle_{L^2}.
\end{aligned}$$

□

We now work towards proving the *Kähler identities*, which say

$$[\overline{\partial}^*, L] = i\partial, \quad [\partial^*, L] = -i\overline{\partial}, \quad [\Lambda, \overline{\partial}] = -i\partial^*, \quad [\Lambda, \partial] = i\overline{\partial}^*.$$

We begin by proving these on $\mathbb{C}^n$ with the standard Kähler metric. In this case we have

$$\omega = \frac{i}{2}\sum_j dz_j \wedge d\overline{z}_j \quad \text{and} \quad g = \frac{1}{2}\sum_j dz_j \otimes d\overline{z}_j.$$

Definition 4.14. *For $\alpha \in \mathscr{A}^k_{\mathbb{C}}$, $\xi \in \mathscr{A}^1_{\mathbb{C}}$, we define the* **cup operator**, *$\xi \vee \alpha \in \mathscr{A}^k_{\mathbb{C}}$ by:*

$$g_{\mathbb{C}}(\xi \vee \alpha, \beta) = g_{\mathbb{C}}(\alpha, \overline{\xi} \wedge \beta) \qquad \forall \beta \in \mathscr{A}^{k-1}_{\mathbb{C}}.$$

This operator exists and is well-defined as $g_{\mathbb{C}}$ is non-degenerate (this is just an exercise in linear algebra to see, e.g. $dz_1 \vee \alpha = \alpha\left(\frac{\partial}{\partial z_1}, \cdot, \cdots, \cdot\right)$).

If $\alpha \in \mathscr{A}^k_{\mathbb{C}}$ then (using multi-index notation) we can write

$$\alpha = \sum_{|I|+|J|=k} \alpha_{IJ} dz_I \wedge d\overline{z}_J$$

and we then define:

$$\partial_j \alpha := \sum_{|I|+|J|=k} \frac{\partial \alpha_{IJ}}{\partial z_j} dz_I \wedge d\overline{Z}_J \quad \text{and} \quad \overline{\partial}_j \alpha = \sum_{|I|+|J|=k} \frac{\partial \alpha_{IJ}}{\partial \overline{z}_j} dz_I \wedge \overline{z}_J$$

so that
$$\mathrm{d}\alpha = \sum_j \Big(\mathrm{d}z_j \wedge \partial_j \alpha + \mathrm{d}\overline{z}_j \wedge \overline{\partial}_j \alpha\Big).$$

Lemma 4.3. *We have* $\mathrm{d}z_j \vee \mathrm{d}z_k = 0$ *and* $\mathrm{d}z_j \vee \mathrm{d}\overline{z}_k = \delta_{jk}$ *for all* j,k*.*

Proof. By definition,
$$\mathrm{d}z_j \vee \mathrm{d}z_k = g_{\mathbb{C}}(\mathrm{d}z_j, \mathrm{d}\overline{z}_k) = 0$$
and
$$\mathrm{d}z_j \vee \mathrm{d}\overline{z}_k = g_{\mathbb{C}}(\mathrm{d}z_j, \mathrm{d}z_k) = \delta_{jk}.$$
□

Lemma 4.4. *We have the following:*

(i) $\overline{\partial}\alpha = \sum_j \mathrm{d}\overline{z}_j \wedge \overline{\partial}_j \alpha$.

(ii) $\partial_j g_{\mathbb{C}}(\alpha,\beta) = g_{\mathbb{C}}(\partial_j \alpha, \beta\rangle + g_{\mathbb{C}}(\alpha, \overline{\partial}_j \beta)$.

(iii) $\partial_j(\mathrm{d}z_k \vee \alpha) = \mathrm{d}z_k \vee \partial_j \alpha$.

Proof. (i): Follows from the definition of $\overline{\partial}$.

(ii): Follows as the metric is the standard one, so:
$$\partial_j g_{\mathbb{C}}(\alpha,\beta) = \partial_j\left(\sum_{I,J} \alpha_{IJ}\overline{\beta}_{IJ}\right) = \sum_{I,J}\Big[\big(\partial_j \alpha_{IJ}\big)\overline{\beta}_{IJ} + \alpha_{IJ}\partial_j \overline{\beta}_{IJ}\Big]$$
and the result then follows, noting that $\partial_j \overline{\beta}_{IJ} = \overline{\overline{\partial}_j \beta_{IJ}}$.

(iii): Follows as ∂_j commutes with $\mathrm{d}z_k\vee$, which follows since it commutes with $\mathrm{d}\overline{z}_k\wedge$. Explicitly, using (ii), for any β:
$$\begin{aligned} g_{\mathbb{C}}\big(\partial_j(\mathrm{d}z_k \vee \alpha), \beta\big) &= \partial_j\big(g_{\mathbb{C}}(\mathrm{d}z_k \vee \alpha, \beta)\big) - g_{\mathbb{C}}\Big(\mathrm{d}z_k \vee \alpha, \overline{\partial}_j \beta\Big) \\ &= \partial_j g_{\mathbb{C}}(\alpha, \mathrm{d}\overline{z}_k \wedge \beta) - g_{\mathbb{C}}\Big(\alpha, \mathrm{d}\overline{z}_k \wedge \overline{\partial}_j \beta\Big) \\ &= g_{\mathbb{C}}\Big(\partial_j \alpha, \overline{\partial}_j(\mathrm{d}\overline{z}_k \wedge \beta)\Big) \qquad \text{using (ii) again} \\ &= g_{\mathbb{C}}\big(\mathrm{d}\overline{z}_k \vee \partial_j \alpha, \beta\big) \end{aligned}$$
and thus as β was arbitrary and $g_{\mathbb{C}}$ is non-degenerate we are done.

□

Lemma 4.5. $\overline{\partial}^* \alpha = -\sum_j \mathrm{d}z_j \vee \partial_j \alpha$.

Proof. Let $\alpha \in \mathscr{A}^k_{\mathbb{C}}$, and let $\beta \in \mathscr{A}^{k-1}_{\mathbb{C}}$ have compact support. Then by Stoke's theorem (where dVol is the standard volume form):

$$\int_{\mathbb{C}^n} \partial_j g_{\mathbb{C}}(\mathrm{d}z_j \vee \alpha, \beta)\,\mathrm{dVol} = 0.$$

Bt we have

$$\begin{aligned} 0 = \int_{\mathbb{C}^n} \partial_j g_{\mathbb{C}}\left(\mathrm{d}z_j \vee \alpha, \beta\right)\,\mathrm{dVol} &= \left\langle \partial_j(\mathrm{d}z_j \vee \alpha), \beta\right\rangle_{L^2} + \left\langle \mathrm{d}z_j \vee \alpha, \overline{\partial}_j \beta\right\rangle_{L^2} \\ &= \left\langle \mathrm{d}z_j \vee \partial_j \alpha, \beta\right\rangle_{L^2} + \left\langle \mathrm{d}z_j \vee \alpha, \overline{\partial}_j \beta\right\rangle_{L^2} \qquad (\ddagger) \end{aligned}$$

using Lemma 4.4(iii). Thus as

$$\langle \overline{\partial}^* \alpha, \beta\rangle_{L^2} = \langle \alpha, \overline{\partial}\beta\rangle_{L^2}$$

as these are L^2-adjoints and since

$$\langle \alpha, \overline{\partial}\beta\rangle_{L^2} = \sum_j \left\langle \alpha, \mathrm{d}\overline{z}_j \wedge \partial_j \beta\right\rangle_{L^2} = \sum_j \left\langle \mathrm{d}z_j \vee \alpha, \overline{\partial}_j \beta\right\rangle_{L^2} = -\sum_j \left\langle \mathrm{d}z_j \vee \partial_j \alpha, \beta\right\rangle_{L^2}$$

by (‡), this gives the result as it shows it holds for all β with compact support.

□

Lemma 4.6. *On $\mathbb{C}^n$ with the standard metric we have*

$$[\overline{\partial}^*, L] = i\partial.$$

Proof. We have by definition of the commutator and of L,

$$[\overline{\partial}^*, L]\alpha = \overline{\partial}^*(L\alpha) - L(\overline{\partial}^*\alpha) = \overline{\partial}^*(\omega \wedge \alpha) - \omega \wedge (\overline{\partial}^*\alpha).$$

Now note that by Lemma 4.5,

$$\begin{aligned} \overline{\partial}^*(\omega \wedge \alpha) &= -\sum_j \mathrm{d}z_j \vee \partial_j(\omega \wedge \alpha) \\ &= -\sum_j \mathrm{d}z_j \vee \left(\partial_j \omega \wedge \alpha + \omega \wedge \partial_j \alpha\right). \end{aligned}$$

But as ω is the standard Kähler form we know $\partial_j \omega = 0$ and thus

$$\begin{aligned} \overline{\partial}^*(\omega \wedge \alpha) &= -\frac{i}{2}\sum_{j,k} \mathrm{d}z_j \vee \left(\mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k \wedge \partial_j \alpha\right) \\ \text{(by Lemma 4.3)} \quad &= -\frac{i}{2}\sum_{j,k}\left[\underbrace{\left(\mathrm{d}z_j \vee \mathrm{d}z_k\right)}_{=0} \wedge \mathrm{d}\overline{z}_k \wedge \partial_j \alpha - \mathrm{d}z_k \wedge \underbrace{\left(\mathrm{d}z_j \vee \mathrm{d}\overline{z}_k\right)}_{=\delta_{jk}} \wedge \partial_j \alpha + \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k \wedge \left(\mathrm{d}z_j \vee \partial_j \alpha\right)\right] \\ &= 0 + \frac{i}{2}\sum_k \mathrm{d}z_k \wedge \partial_k \alpha - \frac{i}{2}\omega \wedge \sum_j \mathrm{d}z_j \vee \partial_j \alpha \\ &= i\partial\alpha + \omega \wedge \overline{\partial}^*\alpha \end{aligned}$$

where we have used Lemma 4.5. Thus we see

$$[\overline{\partial}^*, L]\alpha = i\partial\alpha + \omega \wedge \overline{\partial}^*\alpha - \omega \wedge \overline{\partial}^*\alpha = i\partial\alpha$$

as required. $\square$

Theorem 4.1 (Kähler Identities). *For X a Kähler manifold we have*

(i) $[\overline{\partial}^*, L] = i\partial$

(ii) $[\partial^*, L] = -i\overline{\partial}$

(iii) $[\Lambda, \overline{\partial}] = -i\partial^*$

(iv) $[\Lambda, \partial] = i\overline{\partial}^*$.

Proof. (i): As ω is Kähler, around any $x \in X$ $\exists$ coordinates $z_a, \ldots, z_n$ in which $\omega = \omega_0 + O(|z|^2)$, where ω_0 is the standard metric on $\mathbb{C}^n$. Now as $[\overline{\partial}^*, L]$ only involves the metric and the first derivative of its coefficients, (i) follows from the result on $\mathbb{C}^n$ (Lemma 4.6).

(ii): Follows from (i) by conjugation and using the fact that ω is real.

(iii) + (iv): These follow from (i) and (ii) by taking adjoints.

$\square$

Theorem 4.2. *On a Kähler manifold (X, ω) we have $\Delta_{\mathrm{d}} = 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}$.*

Remark: This is not true on arbitrary complex manifolds.

Proof. We claim that

$$\overline{\partial}^*\partial + \partial\overline{\partial}^* = 0 \qquad \text{and} \qquad \partial^*\overline{\partial} + \overline{\partial}\partial^* = 0.$$

Indeed, the Kähler identities give $\overline{\partial}^* = -i\langle[\Lambda, \partial]$ and so

$$\begin{aligned}\overline{\partial}^*\partial + \partial\overline{\partial}^* &= -i[\Lambda, \partial]\partial - i\partial[\Lambda, \partial]\\ &= i\Lambda\partial\partial + i\partial\Lambda\partial - i\partial\Lambda\partial + i\partial\partial\Lambda\\ &= 0 \quad \text{as } \partial^2 = 0.\end{aligned}$$

Similarly we have $\partial^*\overline{\partial} + \overline{\partial}\partial^* = 0$ (either proof in same way or just take adjoints).

Next we show that $\Delta_{\mathrm{d}} = \Delta_{\partial} + \Delta_{\overline{\partial}}$. Indeed,

$$\begin{aligned}\Delta_{\mathrm{d}} &= \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*\\ &= (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) + (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*)\\ &= \Delta_{\partial} + \Delta_{\overline{\partial}}\end{aligned}$$

as several terms now cancel by the previous observation. Finally we show that $\Delta_\partial = \Delta_{\overline{\partial}}$. By the Kähler identities:

$$\begin{aligned}\Delta_\partial = \partial\partial^* + \partial^*\partial &= i\partial[\Lambda,\overline{\partial}] + i[\Lambda,\overline{\partial}]\partial \\ &= i\partial\Lambda\overline{\partial} - i\partial\overline{\partial}\Lambda + i\Lambda\overline{\partial}\partial - i\overline{\partial}\Lambda\partial\end{aligned}$$

and

$$\begin{aligned}\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} &= -i\overline{\partial}[\Lambda,\partial] - i[\Lambda,\partial]\overline{\partial} \\ &= -i\overline{\partial}\Lambda\partial + i\overline{\partial}\partial\Lambda - i\Lambda\partial\overline{\partial} + i\partial\Lambda\overline{\partial} \\ &= \Delta_\partial\end{aligned}$$

since $\partial\overline{\partial} = -\overline{\partial}\partial$. So we are done.

□

Lemma 4.7. *Let $\alpha \in \mathscr{A}^{p-1,q-1}(X)$, $\beta \in \mathscr{A}^{p,q}(X)$. Then,*

$$g_{\mathbb{C}}(L\alpha,\beta) = g_{\mathbb{C}}(\alpha,\Lambda\beta).$$

Proof. We have

$$\begin{aligned}g_{\mathbb{C}}(L\alpha,\beta)\,\mathrm{dVol} &= (L\alpha)\wedge(*\beta) && \text{by definition of } * \\ &= (\omega\wedge\alpha)\wedge(*\beta) && \text{by definition of } L \\ &= \alpha\wedge\omega\wedge(*\beta) && \text{as } \wedge \text{ is associative and graded commutative} \\ &= g_{\mathbb{C}}(\alpha, *^{-1}L*\beta)\,\mathrm{dVol} && \text{by definition of } * \text{ again} \\ &= g_{\mathbb{C}}(\alpha,\Lambda\beta)\,\mathrm{dVol} && \text{by definition of } \Lambda.\end{aligned}$$

□

Theorem 4.3 (Kähler Identities II). *Let (X,ω) be Kähler. Let $\pi_k : \mathscr{A}^*_{\mathbb{C}} \to \mathscr{A}^k_{\mathbb{C}}$ be the projection, and define*

$$H = \sum_{k=0}^{2n}(n-k)\pi_k \qquad \text{(the } \textbf{counting operator}\text{)}.$$

Then:

(i) H,Λ,L *commute with* Δ_{d}

(ii) $[\Lambda,L] = H$, $[H,L] = -2L$, $[H,\Lambda] = 2\Lambda$.

Proof. We first consider commutators with H. By linearity, it suffices to prove these results for some $\alpha \in \mathscr{A}^{p,q}$, where $p+q=k$.

For such α we have

$$[H,\Delta_{\mathrm{d}}]\alpha = H(\Delta_{\mathrm{d}}\alpha) - \Delta_{\mathrm{d}}(H\alpha) = (n-k)\Delta_{\mathrm{d}}\alpha - (n-k)\Delta_{\mathrm{d}}\alpha = 0$$

and so H commutes with Δ_{d}. Also,
$$[H,L]\alpha = H(L\alpha) - L(H\alpha) = (n-(k+2))L\alpha - L(n-k)\alpha = -2L\alpha$$
i.e.
$$[H,L] = -2L$$
since $L : \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q+1}$. Then taking adjoints and using that $H = H^*$ (i.e. $g_{\mathbb{C}}(H\alpha,\beta) = g_{\mathbb{C}}(\alpha, H\beta)$ always) gives
$$[H,\Lambda] = 2\Lambda.$$
Showing that L commutes with Δ_{d}, i.e. $[L,\Delta_{\mathrm{d}}] = 0$, is equivalent to asking $\Delta_{\mathrm{d}}\omega = 0$ (from the definition of L), i.e. we need to show that ω is harmonic. Showing this is an exercise on Example Sheet 3.

Also since $\Delta_{\mathrm{d}} = \Delta_{\mathrm{d}}^*$ is self-adjoint, we also see $[\Lambda,\Delta_{\mathrm{d}}] = 0$. Thus we have shown H,Λ,L commute with Δ_{d}.

Lastly we need to show $[\Lambda, L] = H$, that is, if $\alpha \in \mathscr{A}^{p,q}$ then:
$$[\Lambda,L]\alpha = (n-p-q)\alpha.$$
This involves no derivatives, and so it holds for (X,ω) if it holds for $\mathbb{C}^n$ w.r.t the standard Kähler metric. We check this explicitly. When $n=1$ we have
$$\Lambda\left(\frac{i}{2}g(z)\mathrm{d}z\wedge \mathrm{d}\overline{z}\right) = g(z)$$
and so the identity holds. In general, write $L = \sum_j L_j$, where $L_j\alpha = \frac{i}{2}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j \wedge \alpha$, and write $\Lambda = \sum_j \Lambda_j$, where $\Lambda_j = L_j^*$ removes $\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ if α has a $\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j$ term and if not $\Lambda_j\alpha = 0$ (up to the appropriate dimensional constant).

Then $[L_j,\Lambda_l] = 0$ if $j \neq l$, and so this reduces to (a small variant of) the one dimensional case, Then by linearity one reduces to
$$\alpha = \frac{i}{2}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j \wedge \hat{\alpha}$$
where $\hat{\alpha} \in \mathscr{A}^{p-1,q-1}$. Then $[\Lambda_j, L_j]\alpha = (n-p-q)\alpha$, as in the one-dimensional case.

□

Remark: See Huybrechts, Proposition 1.2.26 for a proof of Theorem 4.3 which carefully keeps track of the constants.

5. Hodge Theory

We wish to understand the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(X)$ and how they compare with $H^k(X,\mathbb{C})$ (the singular cohomology), where $k=p+q$. We begin by picking canonical representatives of cohomology classes.

Definition 5.1. *Given an oriented Riemannian manifold (X,g), we define the* ***space of harmonic forms of degree k*** *to be:*

$$\mathscr{H}^k(X,g):=\{\alpha\in\mathscr{A}^k(X):\ \Delta_{\mathrm{d}}\alpha=0\}.$$

Each element $\alpha\in\mathscr{H}^k(X,g)$ is called a ***harmonic k-form****.*

Remark: On $\mathbb{R}^n$ with the Euclidean metric, if $f\in C^\infty(\mathbb{R}^n)$, then $\Delta_{\mathrm{d}}f=\Delta f$, where Δ is the usual Laplacian on $\mathbb{R}^n$. So $\Delta_{\mathrm{d}}f=0$ if and only if f is harmonic in the classical sense.

Lemma 5.1. *For (X,ω) a compact Kähler manifold (without boundary always!) we have*

$$\Delta_{\overline{\partial}}\alpha=0 \quad\Longleftrightarrow\quad \overline{\partial}\alpha=\overline{\partial}^*\alpha=0.$$

Proof. $\underline{(\Leftarrow):}$ If $\overline{\partial}\alpha=0=\overline{\partial}^*\alpha$, then $\Delta_{\overline{\partial}}\alpha=0$ by definition of $\Delta_{\overline{\partial}}$.

$\underline{(\Rightarrow):}$ Conversely, if $\Delta_{\overline{\partial}}\alpha=0$, then:

$$\begin{aligned}0=\langle\Delta_{\overline{\partial}}\alpha,\alpha\rangle_{L^2}&=\langle(\overline{\partial}^*\overline{\partial}+\overline{\partial}\,\overline{\partial}^*)\alpha,\alpha\rangle_{L^2}\\&=\langle\overline{\partial}\alpha,\overline{\partial}\alpha\rangle_{L^2}+\langle\overline{\partial}^*\alpha,\overline{\partial}^*\alpha\rangle\\&=\|\overline{\partial}\alpha\|^2_{L^2}+\|\overline{\partial}^*\alpha\|^2_{L^2}\end{aligned}$$

and so we must have $\|\overline{\partial}\alpha\|_{L^2}=0=\|\overline{\partial}^*\alpha\|_{L^2}$, i.e. $\overline{\partial}\alpha=\overline{\partial}^*\alpha=0$.

□

If (X,ω) is Kähler then we can show:

$$\Delta_{\mathrm{d}}\alpha=0\quad\Longleftrightarrow\quad\Delta_{\overline{\partial}}\alpha=0\quad\Longleftrightarrow\quad\Delta_{\partial}\alpha=0.$$

So let

$$\mathscr{H}^{p,q}_{\overline{\partial}}(X,g):=\{\alpha\in\mathscr{A}^{p,q}(X):\ \Delta_{\overline{\partial}}\alpha=0\}.$$

Recall from Differential Geometry the Hodge decomposition:

Theorem 5.1 (Hodge Decomposition for Riemannian Manifolds)**.** *Let (X,g) be a compact oriented Riemannian manifold. Then there is an L^2-orthogonal decomposition*

$$\mathscr{A}^k(X)\cong\mathscr{H}^k(X)\oplus\mathrm{d}\mathscr{A}^{k-1}(X)\oplus\mathrm{d}^*\mathscr{A}^{k+1}(X).$$

In particular the spaces $\mathscr{H}^k(X)$ of harmonic forms are finite dimensional.

Remark: Another way to write this result is:

$$\mathscr{A}^k(X) \cong \mathscr{H}^k(X) \oplus \Delta_{\mathrm{d}}(\mathscr{A}^k(X))$$

since

$$\Delta_{\mathrm{d}}\mathscr{A}^k(X) = \mathrm{dd}^*\mathscr{A}^k(X) \oplus \mathrm{d}^*\mathrm{d}\mathscr{A}^k(X) = \mathrm{d}\mathscr{A}^{k-1}(X) \oplus \mathrm{d}^*\mathscr{A}^{k+1}(X)$$

where we have used the Hodge decomposition above to show $\mathrm{dd}^*\mathscr{A}^k(X) = \mathrm{d}\mathscr{A}^{k-1}(X)$ and $\mathrm{d}^*\mathrm{d}\mathscr{A}^k(X) = \mathrm{d}^*\mathscr{A}^{k+1}(X)$ (e.g. for the first equality, one inclusion is clear, namely $\mathrm{dd}^*\mathscr{A}^k(X) \subset \mathrm{d}\mathscr{A}^{k-1}(X)$ since $\mathrm{d}^*\mathscr{A}^k(X) \subset \mathscr{A}^{k-1}(X)$, and then for the other inclusion suppose $\alpha \in \mathrm{d}\mathscr{A}^{k-1}(X)$, then we can write $\alpha = \mathrm{d}\beta$ for some $\beta \in \mathscr{A}^{k-1}(X)$, and thus from the Hodge decomposition $\beta = \beta_1 + \beta_2 + \beta_3$ where $\beta_1 \in \mathscr{H}^{k-1}(X)$, $\beta_2 \in \mathrm{d}\mathscr{A}^{k-2}(X)$, $\beta \in \mathscr{A}^k(X)$. Then we have $\mathrm{d}\beta = \mathrm{d}\beta_3$, because $\mathrm{d}\beta_2 = 0$ clearly and $\mathrm{d}\beta_1 = 0$ by a similar argument to Lemma 5.1. Thus $\alpha = \mathrm{d}\beta = \mathrm{d}\beta_3 = \mathrm{dd}^*\gamma$ for some $\gamma \in \mathscr{A}^k(X)$ by definition of β_3, i.e. $\alpha \in \mathrm{dd}^*\mathscr{A}^k(X)$ as we wanted).

Proof of Theorem 5.1. None given (uses the theory of elliptic PDEs). □

Theorem 5.2 (Hodge Decomposition for Kähler Manifolds)**.** *Let (X,ω) be a compact Kähler manifold. Then there is an L^2-orthogonal decomposition:*

$$\begin{aligned}\mathscr{A}^{p,q}(X) &\cong \mathscr{H}^{p,q}_{\overline{\partial}}(X) \oplus \overline{\partial}\mathscr{A}^{p,q-1}(X) \oplus \overline{\partial}^*\mathscr{A}^{p,q+1}(X)\\ &\cong \mathscr{H}^{p,q}_{\partial}(X) \oplus \partial\mathscr{A}^{p-1,q}(X) \oplus \partial^*\mathscr{A}^{p+1,q}(X)\end{aligned}$$

where by L^2-orthogonal we mean orthogonal w.r.t the L^2-inner product $\langle\alpha,\beta\rangle_{L^2} = \int_X g_{\mathbb{C}}(\alpha,\beta)\,\mathrm{dVol}$.

Proof. None given. The proof uses techniques from Elliptic PDE theory (see Griffiths-Harris, §0.6 for a discussion). □

Note: We always have

$$\mathscr{H}^{p,q}_{\partial}(X) = \mathscr{H}^{p,q}_{\overline{\partial}}(X) = \mathscr{H}^{p,q}_{\mathrm{d}}(X)$$

since $\Delta_{\mathrm{d}} = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}$ using the Kähler condition (Theorem 4.2).

Corollary 5.1. *Every Dolbeault cohomology group has a unique harmonic representative, i.e. the map $\mathscr{H}^{p,q}_{\overline{\partial}}(X) \to H^{p,q}_{\overline{\partial}}(X)$ sending $\alpha \mapsto \alpha$ is an isomorphism.*

Note: This map is well-defined as $\Delta_{\overline{\partial}}\alpha = 0 \Leftrightarrow \overline{\partial}\alpha = 0 = \overline{\partial}^*\alpha$. In particular if $\Delta_{\overline{\partial}}\alpha = 0$ then $\overline{\partial}\alpha = 0$ and so α does define an element of $H^{p,q}_{\overline{\partial}}(X)$. So what this theorem tells us is that if $\overline{\partial}\alpha = 0$, $\exists\tilde{\alpha}$ with $[\tilde{\alpha}] = [\alpha]$ and $\overline{\partial}\tilde{\alpha} = \overline{\partial}^*\tilde{\alpha} = 0$, i.e. we can change representative to assume wlog $\overline{\partial}^*\alpha = 0$ as well.

Proof. First note that this map exists/is well-defined, since if $\Delta_{\overline{\partial}}\alpha = 0$ then $\overline{\partial}\alpha = 0$.

First we show surjectivity. So let $\alpha \in \mathscr{A}^{p,q}(X)$ satisfy $\overline{\partial}\alpha = 0$. By the Hodge decomposition we may write

$$\alpha = \beta_1 + \overline{\partial}\beta_2 + \overline{\partial}^*\beta_3$$

with β_1 harmonic. Thus we have

$$0 = \overline{\partial}\alpha = 0 + 0 + \overline{\partial}\,\overline{\partial}^*\beta_3.$$

But then this implies

$$0 = \langle \underbrace{\overline{\partial}\,\overline{\partial}^*\beta_3}_{=0}, \beta_3\rangle_{L^2} = \langle \overline{\partial}^*\beta_3, \overline{\partial}^*\beta_3\rangle_{L^2} = \|\overline{\partial}^*\beta_3\|^2_{L^2}$$

and so we need $\overline{\partial}^*\beta_3 = 0$. Thus we actually have

$$\alpha = \beta_1 + \overline{\partial}\beta_2.$$

Hence $[\alpha] = [\beta_1] \in H^{p,q}_{\overline{\partial}}(X)$, with β_1 harmonic. Thus this map is surjective.

We now show injectivity. Suppose $\alpha \in \mathscr{H}^{p,q}_{\overline{\partial}}(X)$ is harmonic with $0 = [\alpha] \in H^{p,q}_{\overline{\partial}}(X)$. Then $\alpha = \overline{\partial}\beta$ for some β. But α is harmonic, and so $0 = \overline{\partial}^*\alpha = 0$, which gives

$$\overline{\partial}^*\overline{\partial}\beta = 0.$$

Hence just as before in the proof using the same L^2-inner product argument we have $\overline{\partial}\beta = 0$, and thus $\alpha = 0$. Thus this map is injective and so is an isomorphism.

□

Corollary 5.2. *THe map $\mathscr{H}^k_{\mathbb{C}}(X) \to H^k_{\mathrm{dR}}(X, \mathbb{C})$, $\alpha \mapsto \alpha$, is an isomorphism. That is, each de Rham cohomology class is represented by a unique harmonic form.*

Proof. Same as Corollary 5.1. □

Remark: The vector space $\mathscr{H}^{p,q}(X) \left(\cong H^{p,q}_{\overline{\partial}}(X)\right)$ admits the following operations:

(i) Conjugation $\alpha \mapsto \overline{\alpha}$ sends harmonic forms to harmonic forms (since, e.g. $\overline{\partial}\,\overline{\alpha} = \overline{(\partial\alpha)}$ by Kähler identities), and hence conjugation induces an isomorphism

$$\mathscr{H}^{p,q}(X) \cong \mathscr{H}^{q,p}(X).$$

This is not true for arbitrary compact complex manifolds as we are using the Kähler identities (namely $\Delta_\partial \alpha = 0 \iff \Delta_{\overline{\partial}}\alpha = 0$) - e.g. it fails for the Höpf surface.

(ii) The Hodge star operator $\alpha \mapsto *\alpha$ sends harmonic forms to harmonic forms, since, e.g. $\partial^*(*\alpha) = -*\partial\alpha = 0$ if α is harmonic, and thus $*$ induces an isomorphism

$$\mathscr{H}^{p,q}_{\overline{\partial}}(X) \cong \mathscr{H}^{n-p,n-q}_{\overline{\partial}}(X).$$

(iii) (Serre Duality) Consider the pairing $\mathscr{H}^{p,q}_{\overline{\partial}}(X) \times \mathscr{H}^{n-p,n-q}_{\overline{\partial}} \to \mathbb{C}$ defined by

$$(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta.$$

Then if $\alpha \neq 0$, then $(\alpha, *\alpha) \mapsto \int_X \alpha \wedge (*\alpha) = \|\alpha\|^2_{L^2} > 0$, and thus this gives an isomorphism:

$$\mathscr{H}^{p,q}_{\overline{\partial}}(X) \cong \left[H^{n-p,n-q}_{\overline{\partial}}(X)\right]^*$$

(RHS is dual space). These isomorphisms induce symmetries and pairings on Dolbeault cohomology groups using the canonical isomorphism $\mathcal{H}^{p,q}_{\overline{\partial}}(X) \cong H^{p,q}_{\overline{\partial}}(X)$ (from Corollary 5.1).

(iv) The map $L : \mathscr{A}^{p,q}(X) \to \mathscr{A}^{p+1,q+1}(X)$, $L(\alpha) = \omega \wedge \alpha$, satisfies $[L, \Delta_{\overline{\partial}}] = 0$ (by the Kähler identities, Theorem 4.3(i)) and thus L descends to a map

$$L : \mathcal{H}^{p,q}_{\overline{\partial}}(X) \to \mathcal{H}^{p+1,q+1}_{\overline{\partial}}(X).$$

Now write $h^{p,q} := \dim\left(H^{p,q}_{\overline{\partial}}(X)\right)$, the dimension of a vector space. This is finite as X is compact. The **Hodge diamond** is a convenient way to represent the above isomorphisms:

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 \iddots & & & \vdots & & & \ddots \\
h^{n,0} & & & \vdots & & & h^{0,n} \\
 \ddots & & & h^{n-1,n-1} & & & \iddots \\
 & h^{n,n-1} & & & & h^{n-1,n} & \\
 & & & h^{n,n} & & &
\end{array}$$

The rows are symmetric by conjugation. The columns are symmetric by the Hodge star operator.

Theorem 5.3. *Let (X, ω) be a compact Kähler manifold. Then there is a decomposition*

$$H^k_{\mathrm{dR}}(X, \mathbb{C}) = H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(X)$$

and this decomposition is independent of the chosen Kähler metric.

Proof. The decomposition is induced by the Hodge decomposition:

$$H^k_{\mathrm{dR}}(X, \mathbb{C}) \cong \mathcal{H}^k_{\overline{\partial}}(X) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\partial}}(X) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(X).$$

Thus we just need to show that this decomposition is independent of the chosen ω. It suffices to show that if $\alpha_1 \in \mathcal{H}^{p,q}_{\overline{\partial}}(X, \omega_1)$, $\alpha_2 \in \mathcal{H}^{p,q}_{\overline{\partial}}(X, \omega_2)$ have $[\alpha_1] = [\alpha_2] = H^{p,q}_{\overline{\partial}}(X)$, <u>then</u> $[\alpha_1] = [\alpha_2]$ in $H^k_{\mathrm{dR}}(X, \mathbb{C})$.

So since $[\alpha_1 - \alpha_2] = 0$ in $H^{p,q}_{\overline{\partial}}(X)$, we have $\alpha_1 = \alpha_2 + \overline{\partial}\gamma$ for some γ. Now since α_1, α_2 are d-harmonic we have

$$\mathrm{d}(\overline{\partial}\gamma) = \mathrm{d}(\alpha_1 - \alpha_2) = 0.$$

Next note that $\overline{\partial}\gamma$ is L^2-orthogonal to $\mathcal{H}^{p,q}(X, \omega)$ be the Kähler Hodge decomposition theorem. Then as $\mathcal{H}^k_{\overline{\partial}}(X, \omega) = \mathcal{H}^k_{\mathrm{d}}(X, \omega)$ (and the latter is independent of ω by definition of harmonic as it only depends on the matrix) we have that $\overline{\partial}\gamma$ is orthogonal to $\mathscr{H}^k_{\mathrm{d}}(X,\omega)$. Thus we have

$$\langle \overline{\partial}\gamma, \mathrm{d}^*\varphi\rangle = \langle \underbrace{\mathrm{d}\overline{\partial}\gamma}_{=0}, \varphi\rangle = 0$$

and so this shows $\overline{\partial}\gamma$ is orthogonal to $\mathrm{d}^*\mathscr{A}^{k+1}$. Thus the Hodge decomposition gives $\overline{\partial}\gamma \in \mathrm{d}\mathscr{A}^{k-1}(X)$, and thus $[\alpha_1] = [\alpha_2]$ in $H^k_{\mathrm{dR}}(X,\mathbb{C})$.

□

Now we move on from Hodge theory and look at vector bundles and curvature.

6. Hermitian Vector Bundles

Let $E \to X$ be a complex vector bundle over a complex manifold X.

Definition 6.1. *We define the* ***complexified k-forms over E****,* $\mathscr{A}^k_{\mathbb{C}}(E)$*, by:*

$$\mathscr{A}^k_{\mathbb{C}}(E)(U) := \mathscr{A}^k_{\mathbb{C}}(U) \otimes C^{\infty}(E)(U)$$

where $C^{\infty}(E)(U)$ *denotes the smooth sections of* E*.*

Then from the splitting $\mathscr{A}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathscr{A}^{p,q}$ we have a splitting

$$\mathscr{A}^k_{\mathbb{C}}(E) = \bigoplus_{p+q=k} \mathscr{A}^{p,q}_{\mathbb{C}}(E).$$

Definition 6.2. *A* ***hermitian metric*** *h on E is a smoothly varying hermitian metric h_x on the fibre E_x over $x \in X$.*

If $e_1, \dots, e_r$ is a local frame for E (so $\text{rank}(E) = r$) then $(h_{jk})_{jk}$, where $h_{jk} = h(e_j, e_k)$, is a hermitian matrix for each $x \in X$, whose coefficients vary smoothly with x. A partition of unity argument produces hermitian metrics on any complex vector bundle.

Exercise: If E, F have hermitian metrics, then $E \oplus F$, $E \otimes F$, E^*, $\Lambda^j E$ all admit natural hermitian metrics.

We now take E to be holomorphic as well.

Proposition 6.1. *There is a natural $\mathbb{C}$ linear operator $\overline{\partial}_E : \mathscr{A}^{p,q}_{\mathbb{C}}(E) \to \mathscr{A}^{p,q+1}_{\mathbb{C}}(E)$ satisfying:*

$$\overline{\partial}_E(\alpha \otimes s) = (\overline{\partial}\alpha) \otimes s + \alpha \otimes (\overline{\partial}_E s) \qquad \forall \alpha \in \mathscr{A}^{p,q}_{\mathbb{C}}(U),\ s \in C^{\infty}(E)(U).$$

Proof. In a local holomorphic frame $(e_1, \dots, e_r)$ we define

$$\overline{\partial}_E(\alpha \otimes e_j) := (\overline{\partial}\alpha) \otimes e_j.$$

To see this is well-defined suppose we have this for one frame $(e'_j)_j$, and let $e_j = \sum_{l=1}^r \varphi_{jl} e'_l$ define another local frame $(e_l)_l$, so that the φ_{jl} are local holomorphic functions. Then

$$\begin{aligned}
\overline{\partial}_E(\alpha \otimes e_j) &= \overline{\partial}_E\left(\alpha \otimes \sum_l \varphi_{jl} e'_l\right) && \text{and by definition of } \overline{\partial}_E(\varphi_{jl} \otimes \alpha), \\
&= \sum_l \varphi_{jl} \overline{\partial}\alpha \otimes e'_l && \text{as the } \varphi_{jl} \text{ are holomorphic so } \overline{\partial}\varphi_{jl} = 0 \\
&= \overline{\partial}\alpha \otimes \sum_l \varphi_{jl} \otimes e'_l \\
&= \overline{\partial}\alpha \otimes e_j
\end{aligned}$$

and so this is true independent of the choice of local frame, and thus it can be defined/extended to all of $\mathscr{A}^{p,q}_{\mathbb{C}}(E)$ as we wanted.

□

Definition 6.3. *A* ***connection*** *of a complex vector bundle is a sheaf morphism*

$$D : \mathscr{A}^0_{\mathbb{C}}(E) \to \mathscr{A}^1_{\mathbb{C}}(E)$$

such that $D(fs) = \mathrm{d}f \otimes s + f\,Ds$ *for all* $f \in C^\infty(U)$, $s \in \mathscr{A}^0(E)(U)$.

Now if $e_1, \ldots, e_r$ is a local frame for E, this gives rise to a connection matrix via:

$$De_j = \sum_l \Theta_{jl} e_l$$

where $\Theta = (\Theta_{jl})_{jl}$ is a matrix of 1-forms (i.e. we can just write De_j in the basis).

Definition 6.4. *Let E be a holomorphic vector bundle. We then define* $D' : \mathscr{A}^0_{\mathbb{C}}(E) \to \mathscr{A}^{1,0}_{\mathbb{C}}(E)$, $D'' : \mathscr{A}^0_{\mathbb{C}}(E) \to \mathscr{A}^{0,1}_{\mathbb{C}}(E)$ *to be the projections of D onto the components* $\mathscr{A}^1_{\mathbb{C}}(E) = \mathscr{A}^{1,0}_{\mathbb{C}}(E) \oplus \mathscr{A}^{0,1}_{\mathbb{C}}(E)$, *i.e.*

$$D = D' + D''.$$

We then say ***D is compatible with the holomorphic structure*** *if* $D'' = \overline{\partial}_E : \mathscr{A}^0_{\mathbb{C}}(E) \to \mathscr{A}^{0,1}_{\mathbb{C}}(E)$.

Proposition 6.2 (Local characterisation of connection and holomorphic structure compatibility)**.** *Let D be a connection on E. Then we have:*

D is compatible with the holomorphic structure $\iff$ *For all local holomorphic frames, the connection matrix $(\Theta_{jl})_{jl}$ is given by a matrix of $(1,0)$-forms.*

Proof. ($\Rightarrow$): Suppose D is compatible. Then the $(0,1)$-part of $(\Theta_{jl})_{jl}$ vanishes, since $De_j = \sum_l \Theta_{jl} e_l$ and the e_l are holomorphic (i.e. vanishing locally gives vanishes globally).

($\Leftarrow$) : Conversely, if $(e_1, \ldots, e_r)$ is a local frame and $\alpha_j \in C^\infty(U)$ then

$$D\left(\sum_j \alpha_j e_j\right) = \sum_j \left(\mathrm{d}\alpha_j \otimes e_j + \alpha_j De_j\right).$$

Then projecting onto the $(0,1)$-part (as the connection matrix only has $(1,0)$-forms and so do not need to worry about De_j term) we get

$$D''\left(\sum_j \alpha_j e_j\right) = \sum_j \overline{\partial}\alpha_j \otimes e_j$$

but this was our local definition of $\overline{\partial}_E$. Thus we are done.

□

Definition 6.5. *Let (E,h) be a hermitian vector bundle. Then we say **D is compatible with h** if:*

$$d[(\alpha,\beta)_h] = (D\alpha,\beta)_h + (\alpha, D\beta)_h \qquad \forall \alpha,\beta \in \mathscr{A}^0(E).$$

Proposition 6.3 (Characterisation of connection and metric compatibility)**.** *Let D be a connection on E and suppose h is a hermitian metric on E. Then:*

D is compatible with h $\iff$ for every unitary frame $(e_1,\ldots,e_r)$ the connection matrix is skew-hermitian, i.e. $\Theta_{jl} = -\overline{\Theta}_{lj}$.

Proof. $\underline{(\Rightarrow):}$ If $(e_1,\ldots,e_r)$ is a unitary frame, then $(e_j,e_l)_h = \delta_{jl}$. Then,

$$\begin{aligned} 0 = d(e_j,e_l)_h &= (De_j,e_l)_h + (e_j,De_l)_h \\ &= \left(\sum_k \Theta_{jk}e_k, e_l\right)_h + \left(e_j, \sum_k \Theta_{lk}e_k\right)_h \\ &= \Theta_{jl} + \overline{\Theta}_{lj} \end{aligned}$$

and thus Θ is skew-hermitian.

$\underline{(\Leftarrow):}$ Conversely suppose $(\Theta_{jl})_{jl}$ is skew-hermitian in any unitary frame. It suffices to show:

$$d(\alpha,\beta)_h = (D\alpha,\beta)_h + (\alpha,D\beta)_h$$

holds locally (as this then implies it holds globally). But by the proof of the $(\Rightarrow)$ direction (just reversed) we see that this does hold when $\alpha,\beta \in \{e_1,\ldots,e_r\}$.

Thus it suffices to show that $d(f\alpha,\beta)_h = (D(f\alpha),\beta)_h + (f\alpha,D\beta)_h$ for any smooth function f (as any such element can be written as a sum of the e_i multiplied by smooth functions). But the LHS of this is:

$$\begin{aligned} d(f\alpha,\beta)_h &= df \otimes (\alpha,\beta)_h + f\,d(\alpha,\beta)_h \\ &= df \otimes (\alpha,\beta)_h + f\left((D\alpha,\beta)_h + (\alpha,D\beta)_h\right) \end{aligned}$$

and the RHS is:

$$\begin{aligned} (D(f\alpha),\beta)_h + (f\alpha,D\beta)_h &= (df\otimes\alpha,\beta)_h + (fD\alpha,\beta)_h + (f\alpha,D\beta)_h \\ &= df \otimes (\alpha,\beta)_h + f(D\alpha,\beta)_h + f(\alpha,D\beta)_h \end{aligned}$$

and thus the two sides agree and we are done.

□

Now we shall see given a hermitian and holomorphic vector bundle (E,h), there is a unique connection which is compatible with both the hermitian metric and the holomorphic structure (compare this with the Levi-Civita connection from Riemannian geometry).

Proposition 6.4. *Let (E, h) be a hermitian and holomorphic vector bundle. Then $\exists!$ connection compatible with both structures.*

Definition 6.6. *This connection in Proposition 6.4 is called the* ***Chern connection****.*

Remark: In practice one typically has a hermitian holomorphic vector bundle, and the Chern connection is the "canonical" extra information.

Proof. We begin with the uniqueness. Let $(e_1, \dots, e_r)$ be a local frame, and set $h_{jk} = h(e_j, e_k)$. Write

$$De_j = \Theta_j^k e_k$$

where we are using summation convention (with upper and lower indices). Then,

$$\begin{aligned} \mathrm{d}h_{jk} = \mathrm{d}h(e_j, e_k) &= (De_j, e_k)_h + (e_j, De_k)_h \\ &= \left(\Theta_j^l e_l, e_k\right)_h + \left(e_j, \Theta_k^l e_l\right)_h \\ &= \Theta_j^l h_{lk} + \overline{\Theta}_k^l h_{jl}. \end{aligned}$$

Now as D is compatible with the holomorphic structure, the matrix (Θ_j^l) is a matrix of $(1,0)$-forms (by Proposition 6.2). So,

$$\partial h_{jk} = \Theta_j^l h_{lk} \quad \text{and} \quad \overline{\partial} h_{jk} = \overline{\Theta}_k^l h_{jl}.$$

Thus $\Theta = \partial h \cdot h^{-1}$, and so Θ is completely determined locally by the hermitian metric, and thus so is D. Hence we have uniqueness.

This also constructs such a connection on each trivialisation. Then by uniqueness, these local connections glue to a connection on all of (E, h) and so we are done.

□

Now we see some more results which have a familiar feel from Differential Geometry:

Lemma 6.1. *If D_1, D_2 are two connections on a complex vector bundle, then $D_1 - D_2$ is linear over $\mathscr{A}_{\mathbb{C}}^0(X)$, and hence $D_1 - D_2$ gives an element of $\mathscr{A}_{\mathbb{C}}^1(\mathrm{End}(E))$, for $\mathrm{End}(E)$ the endomorphism bundle (i.e. $D_1 - D_2 : \mathscr{A}_{\mathbb{C}}(E) \to \mathscr{A}_{\mathbb{C}}^1(E)$).*

Moreover if D is a connection on E and $a \in \mathscr{A}_{\mathbb{C}}^1(\mathrm{End}(E))$, then $D + a$ is also a connection.

Proof. We have

$$(D_1 - D_2)(fs) = fD_1 s - fD_1 s = f(D_1 - D_2)(s)$$

using the definition of a connection as the $\mathrm{d}f \otimes s$ terms cancel out. Thus the linearity over $\mathscr{A}_{\mathbb{C}}^0(X)$ is clear.

Now if $a \in \mathscr{A}^1_{\mathbb{C}}(\mathrm{End}(E))$ then a acts on $\mathscr{A}^0(E)$ by multiplication in the form part and evaluation in the E component $(E \times \mathrm{End}(E) \to E)$. Then,

$$(D+a)(fs) = D(fs) + a(fs) = (\mathrm{d}f \otimes s + f\,Ds) + f\,a(s) = \mathrm{d}f \otimes s + f(D+a)(s)$$

and thus this shows $D+a$ is a connection.

□

Note that a connection D extends to a map $D : \mathscr{A}^p_{\mathbb{C}}(E) \to \mathscr{A}^{p+1}_{\mathbb{C}}(E)$ via:

$$D(\alpha \otimes s) := \mathrm{d}\alpha \otimes s + (-1)^p \alpha \wedge Ds \qquad \text{for } \alpha \in \mathscr{A}^p_{\mathbb{C}}(U),\ s \in C^\infty(E)(U).$$

Definition 6.7. *The **curvature** of the connection D is the map $F_D := D \circ D : \mathscr{A}^0_{\mathbb{C}}(E) \to \mathscr{A}^2_{\mathbb{C}}(E)$.*

Lemma 6.2. *F_D is linear over $C^\infty(X, \mathbb{C})$.*

Proof. For any $f \in C^\infty(X, \mathbb{C})$ and $s \in \mathscr{A}^0_{\mathbb{C}}(E)$ we have:

$$\begin{aligned} F_D(fs) = D(\mathrm{d}f \otimes s + f\,Ds) &= \underbrace{\mathrm{d}^2 f}_{=0} \otimes s \underbrace{- \mathrm{d}f \otimes Ds + \mathrm{d}f \otimes Ds}_{\text{cancel}} + f D^2 s \\ &= f D^2 s \\ &= f F_D(s). \end{aligned}$$

□

Corollary 6.1. *F_D is induced by an element of $\mathscr{A}^2_{\mathbb{C}}(\mathrm{End}(E))$, i.e. the curvature is a matrix of 2-forms.*

Proof. By the above. □

It is often useful to have a local expression for the curvature in terms of the connection matrix. Let $e_1, \ldots, e_r$ be a local frame. Then connection matrix is given by:

$$\Theta e_j = \Theta^k_j e_k$$

where the Θ^k_j are 1-forms. Given a local section $s = s^j e_j$ we have:

$$Ds = \mathrm{d}s^j \otimes e_j + s^j \Theta^k_j e_k.$$

We write this relation as: $D = \mathrm{d} + \Theta$. In this notation we then have

$$\begin{aligned} F_D s = D^2 s &= (d+\Theta)(d+\Theta)s \\ &= \mathrm{d}^2 s + (\mathrm{d}\Theta)s - \Theta(\mathrm{d}s) + \Theta(\mathrm{d}s) + \Theta \wedge \Theta s \\ &= (\mathrm{d}\Theta + \Theta \wedge \Theta)s \end{aligned}$$

i.e. $F_D = \mathrm{d}\Theta + \Theta \wedge \Theta$. However as the next lemma shows, we get more information about F_D when we know more about the structure of the connection.

Lemma 6.3 (Properties of the Curvature). *We have the following:*

(i) If (E,h) is hermitian and D is compatible with h, then
$$h(F_D s_j, s_k) + h(s_j, F_D s_k) = 0$$
i.e. F_D is skew-hermitian.

(ii) If D is compatible with the holomorphic structure (so E holomorphic here), then F_D has no $(0,2)$-component, i.e.
$$F_D \in \mathscr{A}^{2,0}_{\mathbb{C}}(\mathrm{End}(E)) \;\oplus\; \mathscr{A}^{1,1}_{\mathbb{C}}(\mathrm{End}(E)).$$

(iii) If D is the Chern connection, then F_D is a skew-hermitian from in $\mathscr{A}^{1,1}_{\mathbb{C}}(\mathrm{End}(E))$.

Proof. <u>(iii):</u> Comes just from combining (i) and (ii), since if the $(0,2)$-component vanishes being skew-hermitian implies that the $(2,0)$-component also vanishes (since conjugation changes $(2,0) \longleftrightarrow (0,2)$).

<u>(i):</u> The statement is local, so it suffices to prove it locally. Let $(e_1,\ldots,e_r)$ be a local unitary frame, so $D = \mathrm{d} + \Theta$ with $\Theta^\dagger = -\Theta$ ($\dagger$ being the hermitian conjugate). Then we have:

$$\begin{aligned} F_D^\dagger = (\mathrm{d}\Theta + \Theta\wedge\Theta)^\dagger &= (\mathrm{d}\Theta)^\dagger - \Theta^\dagger \wedge \Theta^\dagger \\ &= \mathrm{d}(\Theta^\dagger) - \Theta^\dagger\wedge\Theta^\dagger \\ &= -\mathrm{d}\Theta - \Theta\wedge\Theta \\ &= -F_D \end{aligned}$$

as required.

<u>(ii):</u> We know $D : \mathscr{A}^k_{\mathbb{C}}(E) \to \mathscr{A}^{k+1}_{\mathbb{C}}(E)$ splits as $D = D' + D''$, where $D' : \mathscr{A}^{p,q}_{\mathbb{C}}(E) \to \mathscr{A}^{p+1,q}_{\mathbb{C}}(E)$ and similarly for D''. Then $D'' = \overline{\partial}_E$ by hypothesis. Thus,

$$D\circ D = (D' + \overline{\partial}_E)\circ(D' + \overline{\partial}_E) = \underbrace{D'\circ D'}_{=0} + \underbrace{D'\circ\overline{\partial}_E + \overline{\partial}_E \circ D}_{\text{maps into (1,1)}} + \underbrace{\overline{\partial}_E\circ\overline{\partial}_E}_{=0}$$

and so the $(0,2)$-component vanishes. $\square$

Now if (L,h) is a hermitian holomorphic line bundle and if D is the corresponding Chern connection, the above gives $F_D \in \mathscr{A}^{1,1}_{\mathbb{C}}(\mathrm{End}(L))$ is skew-hermitian, and thus F_D is a real $(1,1)$-form (as L is 1-dimensional over $\mathbb{C}$ so is $\mathrm{End}(L)$). In this we have $\Theta = h^{-1}\partial h = \partial \log(h)$ (from before - see proof of Proposition 6.4) and so

$$F_D = \overline{\partial}\partial \log(h).$$

If $X = \mathbb{P}^n$ and $L = \mathscr{O}(1)$, there is a natural hermitian metric on $\mathscr{O}(-1) \cong \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ arising from the usual hermitian metric on $\mathbb{C}^{n+1}$. This induces a hermitian metric on $L = \mathscr{O}(1)$. Then on

$U_0 = \{[z_0 : \cdots : z_n] \mid z_0 \neq 0\}$ we have the Fubini-Study metric,

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_j |z_j|^2\right)$$

(as $z_0 = 1$ by scaling), which is just $\frac{i}{2\pi}F_D$, where F_D is the curvature corresponding to the natural hermitian metric on $\mathcal{O}(1)$.

Definition 6.8. *We say that a line bundle L is **positive** if there is a hermitian metric h on L such that $\frac{i}{2\pi}F_D$ is a Kähler metric on X (where F_D is the curvature of the Chern connection of L).*

Exercise: (See Example Sheet 4) Show that $\left[\frac{i}{2\pi}F_D\right] \in H^2(X,\mathbb{C})$ is equal to $c_1(L)$, the first Chern class of L (the image of $H^1(X,\mathcal{O}^* \cong \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}))$.

Moreover this exercise shows:

$$L \text{ is positive} \quad \Longleftrightarrow \quad c_1(L) \text{ is a Kähler class.}$$

[Recall that a Kähler class was one such that $\exists \omega \in c_1(L)$ which is Kähler.]

On projective space, the line bundle $\mathcal{O}(1) \to \mathbb{P}^n$ admits a hermitian metric h_{FS} with curvature

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi}F_D = \frac{i}{2\pi}\overline{\partial}\partial\log(h_{\mathrm{FS}})$$

(h_{FS} Fubini-Study) which is Kähler. Thus $\mathcal{O}(1)$ is a positive line bundle.

Now more generally if $\varphi : X \to Y$ is a morphism and $(E,h) \to Y$ is a hermitian and holomorphic vector bundle, then the pullback (φ^*E, φ^*h) is a hermitian and holomorphic vector bundle on X. Moreover, if $E = L$ is a line bundle, then $F_D = \overline{\partial}\partial\log(h)$ and

$$\varphi^*F_D = \varphi^*\overline{\partial}\partial\log(h) = \overline{\partial}\partial\log(\varphi^*h)$$

(as pullbacks commutes with $\partial, \overline{\partial}$ and then by definition of the pullback on functions being evaluation).

Now if X ($\subset \mathbb{P}^n$) is projection and $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion, then we obtain $i^*\mathcal{O}(1) = \mathcal{O}(1)|_X$ on X. Moreover if h_{FS} is the Fubini-Study hermitian metric then $i^*h|_{\mathrm{FS}}$ has curvature $i^*\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}|_X$ (up to a factor of $\frac{i}{2\pi}$). But we showed previously that the restriction of a Kähler metric is also a Kähler metric (see Proposition 4.1) and so $\omega_{\mathrm{FS}}|_X$ is a Kähler metric on X. Thus $\mathcal{O}(1)|_X \hookrightarrow X$ is also positive.

We now turn to the algebro-geometric analogue.

6.1. **Ampleness.**

If X is a compact complex manifold, one cannot embed X in $\mathbb{C}^n$ for any n, as X has no non-constant holomorphic functions (essentially by Liouville's theorem). So instead the idea is to use (holomorphic) sections of line bundles to embed X instead in some $\mathbb{P}^n$.

So let $L \to X$ be a holomorphic line bundle.

Definition 6.9. *A* ***trivialisation*** *of L over $U \subset X$ is a $\xi \in \mathcal{O}^*(L)(U)$, i.e. a nowhere vanishing section (recall $\mathcal{O}^*(L)$ is non-vanishing sections, not the dual!).*

So let $s_0, \ldots, s_n \in H^0(X, L)$ be global sections and suppose for all $x \in X$, there is an s_j with $s_j(x) \neq 0$. Let ξ be a trivialisation of L over some $U \subset X$. So then $s_j = \xi f_j$, for some $f_j \in \mathcal{O}(U)$. Then note for each $x \in X$, for some j we have $s_j(x) \neq 0$, and so as ξ is never zero, we also have $f_j(x) \neq 0$ for some j. Hence:

$$[f_0(x) : \cdots : f_n(x)] \in \mathbb{P}^n$$

as not all the s_j are 0.

We claim that this element of $\mathbb{P}^n$ is independent of the choice of ξ. Indeed, if $\tilde{\xi}$ is another trivialisation then $\tilde{\xi} = g\xi$ for some $g \in \mathcal{O}^*(U)$, and then

$$[f_0(x) : \cdots : f_n(x)] = [g(x)f_0(x) : \cdots : g(x)f_n(x)]$$

and this shows that the point is independent of ξ. We will denote this point by:

$$[s_0 : \cdots : s_n(x)] \in \mathbb{P}^n.$$

Definition 6.10. *We say that L is* ***basepoint free*** *if for all $x \in X$, $\exists s \in H^0(X, L)$ with $s(x) \neq 0$.*

Thus if L is basepoint free, after choosing a basis $(s_i)_{i=0}^n$ of $H^0(X, L)$ we obtain a map $\varphi_L : X \to \mathbb{P}^n$ given by:

$$\varphi_L(x) := [s_0(x) : \cdots : s_n(x)].$$

Definition 6.11. *We say a holomorphic line bundle L is* ***very ample*** *if φ_L is an embedding (for some choice of basis of $H^0(X, L)$).*

Definition 6.12. *We say L is* ***ample*** *if $L^{\otimes k} = \underbrace{L \otimes \cdots \otimes L}_{k \text{ times}}$ is very ample for some $k \in \mathbb{Z}_{>0}$.*

Note: These are independent of the choice of basis: any two bases are related by an element ν of $\mathrm{GL}(n+1)$. Then ν induces a biholomorphism of $\mathbb{P}^n$, and $X \to \nu^* X$ using the two bases.

When L is very ample, using the embedding φ_L we have $\varphi_L^* \mathcal{O}(1) \cong L$ (indeed, if z_0 is viewed as a global section of $\mathcal{O}(1) \to \mathbb{P}^n$, then $\varphi_L^* z_0$ is a global section of L). Hence:

$$L \text{ is very ample} \quad \Longleftrightarrow \quad \exists \text{ an embedding } i : X \hookrightarrow \mathbb{P}^n \text{ with } i^*\mathcal{O}(1) \cong L$$

(this is how ampleness with defined earlier - see Definition 4.1).

Thus L is ample if $L^{\otimes k}$ has enough global sections such that (for $k \gg 0$):

(i) $L^{\otimes k}$ is basepoint free

(ii) $\varphi_{L^{\otimes k}}$ is injective: if $x \neq y \in X$, $\exists s \in H^0(X, L^{\otimes k})$ with $s(x) \neq s(y)$

(iii) $d\varphi_{L^{\otimes k}}$ is injective (the usual thing for an embedding).

(By the inverse function theorem, this is equivalent to X being biholomorphic to a submanifold of $\mathbb{P}^n$).

Lemma 6.4. *If $L \to X$ is ample, then L is positive.*

Proof. If L is ample then $L^{\otimes k}$ is very ample for some $k > 0$, and hence $L^{\otimes k} \cong \varphi^*_{L^{\otimes k}}\mathcal{O}(1)$ with $\varphi_{L^{\otimes k}} : X \hookrightarrow \mathbb{P}^n$ an embedding. Hence $L^{\otimes k}$ is positive, i.e. has a hermitian metric h with curvature $\frac{i}{2\pi}$ being Kähler (just from pulling back that on $\mathbb{P}^n$).

So we just need to show: $L^{\otimes k}$ positive $\Rightarrow L$ positive. Indeed, let ξ be a trivialisation of L over $U \subset X$. Then $\xi^{\otimes k}$ is a trivialisation of $L^{\otimes k}$. Then define a metric on L by:

$$|\xi|_h := |\xi^{\otimes k}|_h^{1/k}.$$

This characterises h, as ξ is a trivialisation (i.e. any other trivialisation differs by a non-zero function, and this is linear in the correct way).

The curvature is $\frac{i}{2\pi}F_D = \frac{i}{2\pi}\overline{\partial}\partial\log(h)$ (as working with line bundles) for h is relative to the curvature $\frac{i}{2\pi}F_{1/k}$ of $h^{1/k}$ by:

$$\frac{i}{2\pi}F_{1/k} = \frac{i}{2\pi}\overline{\partial}\partial\log(h^{1/k}) = \frac{1}{k}\cdot\frac{i}{2\pi}\overline{\partial}\partial\log(h)$$

which is seen clearly in a trivialisation (as the metric is determined by functions). Lastly, note that $\frac{1}{k}\cdot\frac{i}{2\pi}F_D$ is Kähler, so we are done.

□

We are working towards the following result, which is the converse to Lemma 6.4. It tells us exactly when a compact complex manifold is projective.

Theorem 6.1 (The Kodaira Embedding Theorem)**.** *Let X be a compact complex manifold, and $L \to X$ a positive line bundle. Then L is ample.*

Proof. We will build up to this. □

This then has the following important corollary:

Corollary 6.2 (Characterisation of Projective)**.** *Let X be a compact complex manifold. Then:*

$$X \text{ is projective} \iff X \text{ admits a line bundle } L \text{ with } c_1(L) \text{ a Kähler class.}$$

Proof. Follows from Theorem 6.1. □

To prove these, we need to return to the cohomology theory of line bundles, via Hodge theory.

So let (X,ω) be a compact Kähler manifold and let (E,h) be a hermitian holomorphic vector bundle. We then obtain a hermitian metric on $\Lambda^{p,q}T^*X$ through ω and hence on $\Lambda^{p,q}T^*X\otimes E$. We denote this metric by $\langle\cdot,\cdot\rangle$.

h also gives a conjugate linear map (or isomorphism - via the natural pairing) and so h gives $E\cong E^*$ (although this is not an isomorphism in our strict sense earlier).

Definition 6.13. *Define a* ***Hodge star operator*** $\overline{*}_E:\Lambda^{p,q}T^*X\otimes E\to\Lambda^{n-p,n-q}T^*X\otimes E$*, defined by:*

$$\overline{*}_E(\varphi\otimes s):=\overline{*\varphi}\otimes h(s)=*\overline{\varphi}\otimes h(s)$$

We then have $(\alpha,\beta)\mathrm{dVol}=\alpha\wedge(\overline{*}_E\beta)$, where here $\wedge$ means the wedge product on the form part of $\alpha,\overline{*}_E\beta$ and the evaluation $E\otimes E^*\to\mathbb{C}$ on the bundle part (i.e. this is how we define $\wedge$ on elements of $\Lambda^{p,q}T^*X\otimes E$).

Definition 6.14. *We define* $\overline{\partial}^*_E:\mathscr{A}^{p,q}_{\mathbb{C}}(E)\to\mathscr{A}^{p,q-1}_{\mathbb{C}}(E)$ *by:*

$$\overline{\partial}^*_E:=-\overline{*}_E\overline{\partial}_E\overline{*}_E.$$

When $E=\mathscr{O}$ is trivial, these definition agree with the previous Hodge star operator, since then

$$\overline{*}_{\mathscr{O}}(\varphi)=\overline{*\varphi}=*\overline{\varphi}$$

and so

$$\overline{\partial}^*_{\mathscr{O}}(\varphi)=-\overline{*}\overline{\partial}\overline{*}(\varphi)=-\overline{*}\overline{\partial}\ \overline{(*\varphi)}=-\overline{*}\overline{(\partial*\varphi)}=-(*\partial*)\varphi$$

as desired. Then we can define harmonic forms with respect to $\overline{\partial}^*_E$, $\overline{\partial}_E$ analogously to before:

Definition 6.15. *Set* $\Delta_E:=\overline{\partial}^*_E\overline{\partial}_E+\overline{\partial}_E\overline{\partial}^*_E$*. We say that* $\alpha\in\mathscr{A}^{p,q}_{\mathbb{C}}(E)$ *is* ***harmonic*** *if* $\Delta_E\alpha=0$.

We write $\mathscr{H}^{p,q}(X,E):=\{\alpha\in\mathscr{A}^{p,q}_{\mathbb{C}}(E):\Delta_E\alpha=0\}$.

Then $\mathscr{A}^{p,q}_{\mathbb{C}}(E)$ admits an L^2-inner product via:

$$\langle\alpha,\beta\rangle_{L^2}:=\int_X\langle\alpha,\beta\rangle\ \mathrm{dVol}.$$

We now see how we can mimic all the Hodge theory from before to Hodge theory of bundles.

Lemma 6.5. $\overline{\partial}_E^*$ *is the* L^2*-adjoint of* $\overline{\partial}_E$*, and* Δ_E *is self-adjoint. Moreover,*

$$\Delta_E \alpha = 0 \iff \overline{\partial}_E \alpha = 0 \text{ and } \overline{\partial}_E^* \alpha = 0.$$

Proof. Similar the the case before (which was essentially just when E was the trivial bundle $\mathcal{O}$ - see Lemma 4.2 and Lemma 5.1). □

Theorem 6.2 (Hodge Decomposition for Bundles)**.** $\exists$ *an* L^2*-orthogonal decomposition:*

$$\mathscr{A}_{\mathbb{C}}^{p,q}(E) = \mathscr{H}^{p,q}(X,E) \oplus \overline{\partial}_E \mathscr{A}_{\mathbb{C}}^{p,q-1}(E) \oplus \overline{\partial}_E^* \mathscr{A}_{\mathbb{C}}^{p,q+1}(E).$$

Thus $\mathscr{H}^{p,q}(X,E)$ *is finite dimensional.*

Proof. Similar to the case when E is trivial (see Theorem 5.2). □

Definition 6.16. *The* ***Dolbeault cohomology classes with respect to a bundle E*** *are:*

$$H_{\overline{\partial}}^{p,q}(X,E) := \frac{\ker\left(\overline{\partial}_E : \mathscr{A}_{\mathbb{C}}^{p,q}(E) \to \mathscr{A}_{\mathbb{C}}^{p,q+1}(E)\right)}{\text{Image}\left(\overline{\partial}_E : \mathscr{A}_{\mathbb{C}}^{p,q-1}(E) \to \mathscr{A}_{\mathbb{C}}^{p,q}(E)\right)}.$$

Theorem 6.3 (Dolbeault's theorem for bundles)**.** *We have* $H_{\overline{\partial}}^{p,q}(X,E) \cong H^q(X,\Omega^p \otimes E)$*, where* Ω^p *is the sheaf of holomorphic p-forms.*

Proof. Similar to the case when E is trivial (see Theorem 2.2). □

Lemma 6.6. *The natural map* $\mathscr{H}^{p,q}(X,E) \to H_{\overline{\partial}}^{p,q}(X,E)$*,* $\alpha \mapsto \alpha$*, is an isomorphism. Thus any* (p,q)*-Dolbeault class with respect to a bundle has a unique harmonic* (p,q)*-form representing the class. Thus:*

$$\mathscr{H}^{p,q}(X,E) \cong H_{\overline{\partial}}^{p,q}(X,E) \cong H^q(X,\Omega^p \otimes E).$$

Proof. Similar to the case when $E \cong \mathcal{O}$ is the trivial bundle (see Corollary 5.1). Once again note that this map is well-defined because if we have $\Delta_E \alpha = 0$, then $\overline{\partial}_E \alpha = 0$ (from Lemma 6.5).

□

As before this Lemma 6.6 is how we can use Hodge theory to understand Dolbeault cohomology.

Now let D be the Chern connection associated to (E,h). Then in a local holomorphic frame, $D = \mathrm{d}+\Theta$, for Θ a matrix of $(1,0)$-forms.

Proposition 6.5. *Given $x \in X$, $\exists$ a holomorphic frame $(e_j)_j$ and coordinates $(z_l)_l$ such that*

$$\langle e_j(z), e_k(z)\rangle_h = \delta_{jk} + O(|z|^2).$$

Proof. None given - similar to the proof of Proposition 4.3. See Demailly "Complex Analytic and Differential Geometry", Proposition 12.10, Chapter VI. □

Definition 6.17. *The $(e_j)_j$ found in Proposition 6.5 are called a* ***normal frame****.*

Thus for the Chern connection, one can find a holomorphic frame which is orthonormal to first order.

Definition 6.18. *Define $L : \mathscr{A}^{p,q}_{\mathbb{C}}(E) \to \mathscr{A}^{p+1,q+1}_{\mathbb{C}}(E)$ by: for $\varphi \in \mathscr{A}^{p,q}_{\mathbb{C}}(X)$ and $s \in \mathscr{A}^0(E)$:*

$$L(\varphi \otimes s) := (\omega \wedge \varphi)\otimes s = L(\varphi)\otimes s$$

where in the last expression L is the Lefschetz operator from before (Definition 4.12).

Similarly define $\Lambda : \mathscr{A}^{p,q}_{\mathbb{C}}(E) \to \mathscr{A}^{p-1,q-1}_{\mathbb{C}}(E)$ by:

$$\Lambda(\varphi\otimes s) = (\Lambda(\varphi))\otimes s$$

where Λ is the inverse Lefschetz operator as before (Definition 4.12).

i.e. this is just saying we can extend our previous definitions of L,Λ to this case by just acting on the first component.

Recall the Kähler identities: $[\Lambda, L] = (n-(p+q))\mathrm{id}$, $[\Lambda, \overline{\partial}] = -i\overline{\partial}^*$. The first extends directly to bundles, but the second changes as follows:

Lemma 6.7 (Nakano Identity)**.** *Let D be the Chern conection. Then*

$$[\Lambda, \overline{\partial}_E] = -i\left(D^{1,0}\right)^*$$

where $D^{1,0} := -\overline{*}_E D^{1,0}_{E^*}\,\overline{*}_E$ and where $D^{1,0}_{E^*}$ is the projection of the Chern connection on E^* onto the $(1,0)$-component.

Proof. Let $\tau \in \mathscr{A}^{p,q}_{\mathbb{C}}(E)$ be given in a normal frame as:

$$\tau = \sum_j \varphi_j \otimes e_j$$

where $\varphi_j \in \mathscr{A}^{p,q}_{\mathbb{C}}(U)$. Then one checks

$$D\tau = \sum_j \mathrm{d}\varphi_j \otimes e_j + O(|z|)$$

and so

$$\overline{\partial}_E s = D^{0,1}s = \sum_j \overline{\partial}\varphi_j \otimes e_j + O(z)$$

and

$$\left(D^{1,0}\right)^* \tau = \sum_j \partial^*\varphi_j \otimes e_j + O(|z|)$$

as $\overline{*}_E = \overline{*} + O(|z|)$, using that the frame is normal. The result then follows from the usual Kähler identity $[\Lambda, \overline{\partial}] = -i\partial^*$.

□

Remark: Huybrechts' proof of this (which is Lemma 5.2.3 in his book) seems incorrect as it uses a holomorphic orthogonal frame (instead of a normal frame) which in general does not exist.

Lemma 6.8. *$\left(D^{1,0}\right)^*$ is the L^2-adjoint of $D^{1,0}$, i.e.*

$$\left\langle \left(D^{1,0}\right)^* \alpha, \beta \right\rangle_{L^2} = \left\langle \alpha, D^{1,0}\beta \right\rangle_{L^2}.$$

Proof. Similar to the case when $E \cong \mathscr{O}$ is trivial. □

Lemma 6.9. *Let $\alpha \in \mathscr{H}^{p,q}(X,E)$ be harmonic. Then:*

(i) $i\langle (F_D\Lambda)(\alpha), \alpha\rangle_{L^2} \leq 0$

(ii) $i\,\langle (\Lambda F_D)(\alpha), \alpha\rangle_{L^2} \geq 0$.

Proof. (i): We have $\Lambda\alpha \in \mathscr{A}^{p-1,q-1}_{\mathbb{C}}(E)$ and so $F_D\Lambda\alpha \in \mathscr{A}^{p,q}_{\mathbb{C}}(X,E)$, and so the statement makes sense. Here F_D acts on α by wedging in the form part and evaluation $\mathrm{End}(E) \times E \to E$ on the bundle part.

As D is the Chern connection, $F_D = D^{1,0}\overline{\partial}_E + \overline{\partial}_E D^{1,0}$. But α is harmonic and so $\overline{\partial}_E\alpha = 0$, $\overline{\partial}^*_E\alpha = 0$. Thus

$$\begin{aligned}
i\langle F_D\Lambda\alpha, \alpha\rangle_{L^2} &= i\left\langle D^{1,0}\overline{\partial}_E\Lambda\alpha, \alpha\right\rangle_{L^2} + i\left\langle \overline{\partial}_E D^{1,0}\Lambda\alpha, \alpha\right\rangle_{L^2} \\
&= -\left\langle \overline{\partial}_E\Lambda, i\left(D^{1,0}\right)^*\alpha\right\rangle_{L^2} + \underbrace{i\left\langle D^{1,0}\Lambda\alpha, \overline{\partial}^*_E\alpha\right\rangle_{L^2}}_{=0 \text{ as } \overline{\partial}^*_E\alpha = 0} \\
&= \left\langle \overline{\partial}_E\Lambda\alpha, [\Lambda, \overline{\partial}_E]\alpha\right\rangle_{L^2} \qquad \text{by Nakano identity} \\
&= -\|\overline{\partial}_E\Lambda\alpha\|^2_{L^2} \leq 0 \qquad \text{as } \overline{\partial}_E\alpha = 0
\end{aligned}$$

where in the second line the negative sign on the first term comes from the inner product being conjugate linear in the second component.

(ii): Similarly we have

$$\begin{aligned}\langle i\Lambda F_D\alpha,\alpha\rangle_{L^2} &= i\left\langle [\Lambda,\overline{\partial}_E]D^{1,0}\alpha,\alpha\right\rangle_{L^2} + i\left\langle \overline{\partial}_E\Lambda D^{1,0}\alpha,\alpha\right\rangle_{L^2} && \text{using that } \overline{\partial}_E\alpha=0\\ &= i\left\langle -i\left(D^{1,0}\right)^* D^{1,0}\alpha,\alpha\right\rangle_{L^2} + \underbrace{i\left\langle \Lambda D^{1,0}\alpha,\overline{\partial}^*_E\alpha\right\rangle}_{=0 \text{ as } \alpha^*_E\alpha=0} && \text{using the Nakano identity}\\ &= \|D^{1,0}\alpha\|^2_{L^2}\\ &\geq 0\end{aligned}$$

and so we are done. □

Theorem 6.4 (Kodaira Vanishing Theorem). *Let L be positive. Then for $p+q>n$ we have*

$$H^q(X,\Omega^p\otimes L)=\{0\}.$$

Proof. Since L is positive we can find a hermitian metric h on L such that $\frac{i}{2\pi}F_D$ is Kähler. Thus $L\alpha=\frac{i}{2\pi}F_D\wedge\alpha$.

So let $\alpha\in\mathscr{H}^{p,q}(X,L)$. Then $[\Lambda,L]=-H$. So by Lemma 6.9,

$$0\leq\left\langle\frac{i}{2\pi}[\Lambda,F_D]\alpha\ ,\ \alpha\right\rangle_{L^2}=\langle[\Lambda,L]\alpha]=,\alpha\rangle_{L^2}=(n-(p+q))\|\alpha\|^2_{L^2}$$

and thus if $p+q>n$ we must have $\alpha=0$. Since $\mathscr{H}^{p,q}(X,L)=\{0\}$ here. But then we are done as $\mathscr{H}^{p,q}(X,L)\cong H^q(X,\Omega^p\otimes L)$ by Lemma 6.6.

□

Another useful vanishing theorem is "Serre vanishing":

Theorem 6.5 (Serre Vanishing). *Suppose $E\to X$ is a holomorphic vector bundle, and suppose $L\to X$ is positive. Then for all k sufficiently large we have:*

$$H^j(X,E\otimes L^{\otimes k})=\{0\}.$$

Proof. The proof uses similar techniques to the Kodaira vanishing theorem. □

6.2. **Blow Ups.**

Definition 6.19. *The **blow-up** of a complex manifold X **at a point $p\in X$** is a complex manifold $\pi:\mathrm{Bl}_pX\to X$ with $\pi^{-1}(p)\cong\mathbb{P}^{n-1}=:E$ a divisor and $\pi:\left(\mathrm{Bl}_pX\right)\backslash E\to X\backslash\{p\}$ an isomorphism.*

Let Δ be the unit disc in $\mathbb{C}^n$. Let $z_1,\dots,z_n$ be coordinates on $\mathbb{C}^n$ and $l=[l_1:\cdots:l_n]$ homogeneous coordinates on $\mathbb{P}^{n-1}$.

Define then:

$$\mathrm{Bl}_0\Delta := \{(z,l) : z_j l_k = z_k l_j \text{ for all } j,k\} \subset \Delta \times \mathbb{P}^{n-1}.$$

This consists of pairs (z,l) with $z\in l$, since $z\in l$ if and only if $z\wedge l=0$ (the wedge product of vectors in $\mathbb{C}^n$, i.e. this just says z is parallel to the line l if and only if the wedge product $z\wedge l$ vanishes).

If one replaces Δ with $\mathbb{C}^n$, this is exactly how we constructed $\mathcal{O}(-1)\to\mathbb{P}^{n-1}$, and so as $\mathcal{O}(-1)$ is a complex manifold, we see by the same reasoning that $\mathrm{Bl}_0\Delta$ is a complex manifold.

The map $\pi:\mathrm{Bl}_0\Delta\to\Delta$ is given by $(z,l)\mapsto z$. Noting that a non-zero point z is contained in a unique line, we have that $\pi:\mathrm{Bl}_0\backslash\{\pi^{-1}(0)\}\to\Delta\backslash\{0\}$ is an isomorphism. Moreover $\pi^{-1}(0)$ consists of all lines, and so is isomorphic to $\mathbb{P}^{n-1}$.

Now more generally, let X be a complex manifold and $p\in X$. Then we know we can find local coordinates so that $z:U\to\Delta\subset X$ is a biholomorphism, i.e. a neighbourhood of p in X is biholomorphic to a disc in $\mathbb{C}^n$. The restriction $\pi:\mathrm{Bl}_p\Delta\backslash E\to\Delta\backslash\{p\}$ gives an isomorphism between a neighbourhood of E in $\mathrm{Bl}_p\Delta$ and of p in X. So we can construct Bl_pX as:

$$\mathrm{Bl}_pX = (X\backslash\{p\})\bigcup_{\pi}\mathrm{Bl}_p\Delta$$

i.e. it is obtained by replacing p with $\mathrm{Bl}_p\Delta$. One obtains $\pi:\mathrm{Bl}_pX\to X$ with the desired properties.

Definition 6.20. *We call $E=\pi^{-1}(p)\cong\mathbb{P}^{n-1}$ the **exceptional divisor**.*

We will quickly show that the exceptional divisor E is independent of the choice of coordinates on Δ (this was z above).

So let $(z'_j)_j$, $z'_j=f_j(z)$ be another choice of coordinates with $f_j(0)=0$ (so the origin/ image of p is preserved). Let $\widetilde{\mathrm{Bl}_0\Delta}$ be the blow up in these new coordinates. Then the isomorphism

$$f:\mathrm{Bl}_p\Delta\backslash E\xrightarrow{\cong}\widetilde{\mathrm{Bl}_p\Delta}\backslash\tilde{E}$$

extends to an isomorphism $f:\mathrm{Bl}_p\Delta\to\widetilde{\mathrm{Bl}_p\Delta}$ by setting $f(0,l):=(0,\tilde{l})$, where

$$\tilde{l}_j=\sum_k\frac{\partial f_k(0)}{\partial z_j}l_k$$

[**Exercise** to check the details of this]. Similarly the identification $E\cong\mathbb{P}\left(T_pX^{(1,0)}\right)$, sending $(0,l)\mapsto\sum_j l_j\frac{\partial}{\partial z_j}$, is independent of the choice of coordinates (i.e. the intuition here is that for the blow up at p we simply replace p by all the tangent vectors at p). Thus as this f is an isomorphism we see the blow up is independent of the choice of coordinates z.

□

Now let $\mathcal{O}(E)$ be the line bundle associated to the divisor E. Then $\mathcal{O}(E)$ can be identified with $\bigcup_{(z,l)}l\to\mathrm{Bl}_p\Delta$ (LHS being a disjoint union as we want to count all zeros in the lines separately), as this admits a section

$$t(z,l) = ((l,z),z)$$

which vanishes along E with multiplicity one. Thus $\mathcal{O}(E) \cong p^*(\mathcal{O}(-1))$ where $p : \mathrm{Bl}_p\Delta \to \mathbb{P}^{n-1}$ is the natural projection, since $\mathrm{Bl}_p\Delta \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$. It follows that $\mathscr{E}|_E \cong \mathcal{O}(-1)$, which is then true for any complex manifold.

Now $\mathcal{O}(E)^* \cong \mathcal{O}(-E)$ has fibre over $(z,l) \in \mathrm{Bl}_p\Delta$ given by the space of linear functionals on the line $l \subset \mathbb{C}^n$, and so $\mathcal{O}(-E)|_E$ is the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}$. Then as $E \cong \mathbb{P}\left(T_pX^{(1,0)}\right)$ we get an isomorphism

$$(\ddagger) \qquad H^0\left(E, \mathcal{O}(-E)|_E\right) \cong T_p^*X^{(1,0)}.$$

Now if $f \in \mathcal{O}(\Delta)$ vanishes at p ($=0$ in coordinate chart), the function π^*f vanishes along E, and so it can be considered as a section of $\mathcal{O}(-E) \to \mathrm{Bl}_p\Delta$. The isomorphism ($\ddagger$) is then

$$H^0(E, \mathcal{O}(E)|_E) \xrightarrow{\cong} T_p^*\Delta^{(1,0)} \qquad \text{sending} \qquad \pi^*f \longmapsto \mathrm{d}f_p.$$

Thus the diagram:

$$\begin{tikzcd}
H^0\left(\mathrm{Bl}_p\Delta, \mathcal{O}(-E)\right) \arrow[r, "r_E"] & H^0\left(E, \mathcal{O}(-E)|_E\right) \\
H^0(\Delta, \mathscr{I}_p) \arrow[u, "\text{pullback}"] \arrow[r, "\mathrm{d}_p"] & T_p^*X^{(1,0)} \arrow[u, "\cong"]
\end{tikzcd}$$

commutes, where r_E is the restriction to E map. Here $\mathscr{I}_p$ is the **ideal sheaf** of p, i.e.

$$\mathscr{I}_p(U) := \{f \in \mathcal{O}(u) : f(p) = 0\}$$

i.e. the sheaf of holomorphic functions vanishing at p.

Proposition 6.6. *Let F be any line bundle on X and let $L \to X$ be positive. Then for any integers $d_1, \dots, d_l > 0$, for all k sufficiently large the line bundle*

$$\pi^*\left(L^{\otimes k} \otimes F\right) \otimes \mathcal{O}\left(-\sum_j d_jE_j\right)$$

is positive on $\mathrm{Bl}_{p_1,\dots,p_l}X$, where the E_j are the exceptional divisors corresponding to blowing up at p_j.

In particular this is true for $F = \mathcal{O}$ the trivial line bundle.

Proof. In a neighbourhood $p_j \in U_j \subset X$, the blow up is $\mathrm{Bl}_{p_j}U_j \subset U_j \times \mathbb{P}^{n-1}$, and

$$\mathcal{O}(E_j) \cong p_j^*(\mathcal{O}(-1)).$$

We give $\mathcal{O}(E_j)$ a metric via the pullback of the Fubini-Study metric on $\mathbb{P}^{n-1}$. Using a partition of unity, this produces metrics (by tensor products) on $\mathcal{O}\left(\sum_j -d_jE_j\right)$.

Locally near E_j, the curvature is $-d_j(2\pi i)p_j^*\omega_{\mathrm{FS}}$. Thus this metric is strictly positive on E_j (i.e. on vectors tangent to E_j) and semi-positive locally. So let $\frac{i}{2\pi}F_D$ be the curvature, and let ω be the curvature of a positive metric on L (ω Kähler in particular). Let α be the curvature of a metric on F.

Then $\pi^*\omega$ is trivial along the exceptional divisor E, and positive everywhere else. Thus

$$\pi^*(k\omega+\alpha)+\frac{i}{2\pi}F_D$$

is Kähler for all k sufficiently large, and this is the curvature of a metric on the desired line bundle, which shows that it is positive.

□

Exercise: For a complex manifold X set $K_X := \Lambda^n T^*X^{(1,0)}$. Then show that

$$K_{\mathrm{Bl}_p X} \cong \pi^* K_X \otimes \mathscr{O}((-n+1)E).$$

6.3. **The Kodaira Embedding Theorem.**

Theorem 6.6 (Hartog's Extension Theorem). *Let $U \subset \mathbb{C}^n$ be open, and $n \geq 2$. Let $f : U\backslash\{z_1 = z_2 = 0\} \to \mathbb{C}$ be holomorphic. Then $\exists$ a unique holomorphic extension $\tilde{f} : U \to \mathbb{C}$ of f.*

Proof. None given - see Huybrechts Proposition 2.16. □

Exercise: Let $L \in \mathrm{Pic}(X)$ and let $Y \subset X$ be a submanifold of codimension ≥ 2. Then show that the restriction $H^0(X,L) \to H^0(X\backslash Y, L)$ is an isomorphism. [This is just a geometric form of Hartog's extension theorem.]

Now onto one of the main results of this course, which we mentioned earlier:

Theorem 6.7 (The Kodaira Embedding Theorem). *If X is a compact complex manifold and $L \to X$ is positive, then L is ample.*

Proof. Let $N_k + 1 = \dim\big(H^0(X, L^{\otimes k})\big)$. We need to show that $\exists k > 0$ such that

(i) For all $x \in X$, $\exists s \in H^0(X, L^{\otimes k})$ with $s(x) \neq 0$.

(ii) For all $x, y \in X$, $\exists$ a section $s \in H^0(X, L^{\otimes k})$ with $s(x) \neq s(y)$.

(iii) For all $x \in X$, $\mathrm{d}(\varphi_{L^{\otimes k}})_x : T_xX \to T_{\varphi_{L^{\otimes k}}(x)}\mathbb{P}^{N_k}$ is injective, where $\varphi :_{L^{\otimes k}}: X \to \mathbb{P}^{N_k}$ is the map

$$\varphi_{L^{\otimes k}}(x) := [s_0(x) : \cdots s_{N_k}(x)] \in \mathbb{P}^{N_k}$$

defined previously (recall Definition 6.11).

So let $L_x^{\otimes k}$ be the fibre of $L^{\otimes k}$ at $x \in X$. (i) asks for $H^0(X, L^{\otimes k}) \to L_x^{\otimes k}$ to be surjective. We know that there is a short exact sequence

$$0 \longrightarrow L^{\otimes k} \otimes \mathscr{I}_x \longrightarrow L^{\otimes k} \longrightarrow L_x^{\otimes k} \longrightarrow 0$$

where $L^{\otimes k} \otimes \mathscr{I}_x$ denotes sections of $L^{\otimes k}$ which vanish at x. Then from the corresponding long exact sequence in cohomology we see that ψ is surjective if $H^1(X, L^{\otimes k} \otimes \mathscr{I}) = \{0\}$.

Similarly the short exact sequence

$$0 \longrightarrow L^{\otimes k} \otimes \mathscr{I}_{x,y} \longrightarrow L^{\otimes k} \longrightarrow L_x^{\otimes k} \otimes L_y^{\otimes k} \longrightarrow 0$$

is related to (ii).

We will prove (ii), and then (i) is very similar. We pass from points to divisors (and hence line bundles) by blowing-up. We have lots of vanishing theorems for divisors, so this is where we are heading to try and prove these cohomology groups vanish.

Let $\tilde{X}$ be the blow-up of X at x and y, with exceptional divisors E_x, E_y. Set $E = E_x + E_y$, and let $\tilde{L} = \pi^* L$, where $\pi : \tilde{X} \to X$ is the natural map (if $\dim(X) = 1$ then we set $\pi = \text{id}$ and $\tilde{X} = X$ as the divisors are just points and so we do not need to blow up points to divisors as they are already divisors).

Consider the pullback $\pi^* : H^0(X, L^{\otimes k}) \to H^0(\tilde{X}, \tilde{L}^{\otimes k})$, which is injective. But then any $\sigma \in H^0(\tilde{X}, \tilde{L}^{\otimes k})$ induces a section $\sigma \in H^0(X \backslash \{x, y\}, L^{\otimes k})$, since $X \backslash \{x, y\} \cong \tilde{X} \backslash E$, inducing $\sigma \in H^0(X, L^{\otimes k})$. Thus in fact π^* is an isomorphism. [We have used Hartog's theorem here].

By construction, $\tilde{L}^{\otimes k}$ is trivial along E_x, E_y, i.e.

$$\tilde{L}^{\otimes k}|_{E_x} \cong E_x \times L_x^{\otimes k} \quad \text{and} \quad \tilde{L}^{\otimes k}|_{E_y} \cong E_y \times L_y^{\otimes k}.$$

So $H^0(E, \tilde{L}^{\otimes k}|_E) \cong L_x^{\otimes k} \oplus L_y^{\otimes k}$. Then if r_E is the restriction $H^0(\tilde{X}, \tilde{L}^{\otimes k}) \to H^0(E, \tilde{L}^{\otimes k}|_E)$ we have that the following diagram

$$\begin{array}{ccc} H^0(\tilde{X}, \tilde{L}^{\otimes k}) & \xrightarrow{r_E} & H^0(E, \tilde{L}^{\otimes k}|_E) \\ \uparrow & & \cong\uparrow \\ H^0(X, L^{\otimes k}) & \xrightarrow{r_{xy}} & L_x^{\otimes k} \oplus L_y^{\otimes k} \end{array}$$

commutes. Thus it suffices to show that r_E is surjective to obtain (ii). So choose k such that

$$\begin{aligned} L' &:= \tilde{L}^{\otimes k} \otimes K_{\tilde{X}}^* \otimes \mathscr{O}(-E) \\ &\cong \pi^*\left(L^{\otimes k} \otimes K_X^*\right) \otimes \mathscr{O}(-nE) \end{aligned}$$

is positive (can do this by Proposition 6.6: here $n = \dim(X)$). Then by the Kodaira vanishing theorem (Theorem 6.4) we have

$$H^1(\tilde{X}, \tilde{L}^{\otimes k} \otimes \mathscr{O}(-E)) = H^1(\tilde{X}, L' \otimes K_{\tilde{X}}) = \{0\}.$$

So considering:

$$0 \longrightarrow \tilde{L}^{\otimes k} \otimes \mathscr{O}(-E) \longrightarrow \tilde{L}^{\otimes k} \xrightarrow{r_E} \tilde{L}^{\otimes k}|_E \longrightarrow 0$$

we see that $r_E : H^0(\tilde{X}, \tilde{L}^{\otimes k}) \to H^0(E, \tilde{L}^{\otimes k}|_E)$ is surjective, which proves (ii) near x, y.

So now we can use compactness to get a global statement. If $\varphi_{L^{\otimes k}}$ is defined at x, y and $\varphi_{L^{\otimes k}}(x) \neq \varphi_{L^{\otimes k}}(y)$, then the same is true for nearby points (by continuity). Thus we get an open cover of X and so by compactness a finite open cover, and then we can apply the above (choosing a k on each of the finitely many sets of the cover) to see that we can find N such that for all $k \geq N$ we have $L^{\otimes k}$ is basepoint free and injective, and so (ii) is established.

We now prove (iii). Let $\varphi_\alpha : U_\alpha \times \mathbb{C} \to L^{\otimes k}|_{U_\alpha}$ be a trivialisation of $L^{\otimes k}$. Then

$$\mathrm{d}(\varphi_{L^{\otimes k}})_x : T_x X \to T_{\varphi_{L^{\otimes k}}(x)}\mathbb{P}^{N_k} \quad \text{is injective} \quad \Longleftrightarrow \quad \begin{array}{l} \forall v^* \in T_x^* X^{(1,0)},\ \exists s \in H^0(X, L^{\otimes k}) \text{ with} \\ s_\alpha = \varphi_\alpha^* s,\ s(x) = 0,\ \mathrm{d}s_\alpha(x) = v^* \end{array}$$

(here we view $\varphi_{L^{\otimes k}}$ locally as (for $s_0(x) \neq 0$) a map $X \to \mathbb{C}^{N_k}$, $y \mapsto (s_1(y), \dots, s_{N_k}(y))$).

More intrinsically, let $L^{\otimes k} \otimes \mathscr{I}_x$ be as before. Then if $s \in L^{\otimes k} \otimes \mathscr{I}_k(U)$, $\varphi_\alpha, \varphi_\beta$ are trivialisations of $L^{\otimes k}$ over U, and if $s_\alpha = \varphi_\alpha^* s$, $s_\beta = \varphi_\beta^* s$, then we have

$$s_\alpha = \varphi_{\alpha\beta} s_\beta, \quad \mathrm{d}s_\alpha = \mathrm{d}s_\beta \cdot \varphi_{\alpha\beta} + \underbrace{\mathrm{d}\varphi_{\alpha\beta} \cdot s_\beta}_{=0 \text{ as } s_\beta(x)=0} = \mathrm{d}s_\beta \cdot \varphi_{\alpha\beta}$$

which gives rise to a sheaf morphism $\mathrm{d}_x : L^{\otimes k} \otimes \mathscr{I}_x \to T_x^* X^{(1,0)} \otimes L_x^{\otimes k}$ (extra $L_x^{\otimes k}$ from $\varphi_{\alpha\beta}$). Then (iii) states that

$$\mathrm{d}_x : H^0(X, L^{\otimes k} \otimes \mathscr{I}_x) \to T^* x X^{(1,0)} \otimes L_x^{\otimes k} \quad \text{is surjective for all } x \in X$$

(or equivalently $H^1(X, L^{\otimes k} \otimes \mathscr{I}_x^2) = \{0\}$ for all $x \in X$).

Now if $\sigma \in H^0(X, L^{\otimes k})$ then $\sigma(x) = 0$ if and only if $\pi^*\sigma = \tilde{\sigma}$ vanishes along E (by Hartog's extension theorem), where $\tilde{X} = \mathrm{Bl}_x X$ (i.e. $\tilde{\sigma}$ corresponds to the blow up at x). Thus π^* induces an isomorphism

$$H^0(X, L^{\otimes k} \otimes \mathscr{I}_x) \to H^0(\tilde{X}, \tilde{L}^{\otimes k} \otimes \mathscr{O}(-E)).$$

We can identify

$$\begin{aligned} H^0\left(E, \left(\tilde{L}^{\otimes k} \otimes \mathscr{O}(-E)\right)\big|_E\right) &= L_x^{\otimes k} \otimes H^0(E, \mathscr{O}(-E)|_E) \\ &= L_x^{\otimes k} \otimes T_x^* X^{(1,0)} \end{aligned}$$

as $\tilde{L}^{\otimes k}|_E$ is trivial. Moreover the diagram

$$\begin{array}{ccc} H^0(\tilde{X}, \tilde{L}^{\otimes k} \otimes \mathscr{O}(-E)) & \xrightarrow{r_E} & H^0\left(E, \left(\tilde{L}^{\otimes k} \otimes \mathscr{O}(-E)\right)\big|_E\right) \\ \pi^* \big\uparrow \cong & & = \big\uparrow \mathrm{id} \\ H^0(X, L^{\otimes k} \otimes \mathscr{I}_x) & \xrightarrow{\mathrm{d}_x} & L_x^{\otimes k} \otimes T_x^* X^{(1,0)} \end{array}$$

commutes. Hence to prove d_x is surjective, it is enough to prove that r_E (the restriction map) is surjective.

So taking k sufficiently large such that $H^1(\tilde{X}, \tilde{L}^{\otimes k} \otimes \mathscr{O}(-2E)) = \{0\}$ (by positivity and Kodaira vanishing), r_E is surjective. Then one obtains a k which works for all $x \in X$ by compactness, just as before. Hence we are done.

□

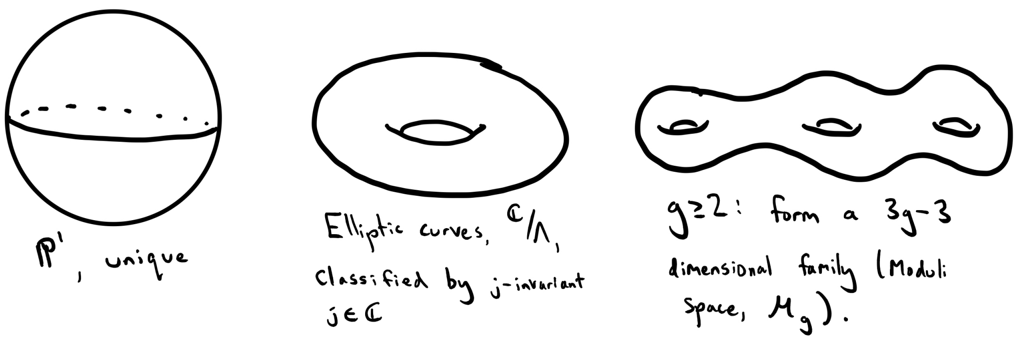
7. Classification of Compact Complex Surfaces

Remark: The book by Beauville is the recommended reference for the contents of this section and for the classification of surfaces.

Definition 7.1. *A **Riemann surface** is a compact complex manifold of dimension 1.*

Let S be a Riemann surface. Then any $(1,1)$-form on S is closed (as $\dim_{\mathbb{R}}(S) = 2$ so higher cohomology groups vanish) and so S is Kähler, and $H^2(S,\mathbb{Z}) = \mathbb{Z}$. Let $\alpha \in H^2(S,\mathbb{Z})$ be Kähler with $\alpha = c_1(L)$, So by Kodaira embedding we know L is ample and so by Corollary 6.2 S is projective.

By Riemann-Roch, a line bundle $L \to S$ is ample $\iff \deg(L) = \int_S \omega = \int_S c_1(L) > 0$ for $\omega \in c_1(L)$. Then Riemann surfaces are classified by their genus g:



Then:

- For $\mathbb{P}^1$, $K^*_{\mathbb{P}^1}$ is ample, $\mathcal{O}(-1), c_1(S) = c_1(K^*_S)$ is Kähler.
- For elliptic curves $K_S \cong \mathcal{O}_S$ and $c_1(S) = 0$.
- For genus $g \geq 2$, K_S is ample and $c_1(S)$ is Kähler.

We wish to see if we can get similar classifications in higher dimensions.

7.1. **Enriques-Kodaira Classification of Surfaces.**

Let X be a compact surface. Set:

$$L_1 \cdot L_2 = \int_X \omega_1 \wedge \omega_2 = \int_X c_1(L_1) \smile c_1(L_2)$$

for $\omega_1 \in c_1(L_1), \omega_2 \in c_1(L_2)$. Then if $\mathcal{O}(D) \cong L_1$, so $Z(s) = D$ for some $s \in H^0(X, L_1)$, then

$$D \cdot L_2 = \int_X \omega_1 \wedge w_2 = \int_D c_1(L_2) = \int_D \omega_2.$$

If $E \subset \mathrm{Bl}_p X$ is the exception divisor, then

$$E \cdot E = \int_E \mathcal{O}(E)|_E = \int_{\mathbb{P}^1} \mathcal{O}(-1) = -1.$$

Given X, we can blow-up to get $\mathrm{Bl}_p X$, a new compact complex surface. Conversely, the following result tells us how to invert a blow up:

Theorem 7.1 (Castelnuovo). *If $\mathbb{P}^1 \cong C \subset X$ has $C \cdot C = -1$, then $\exists Y$ with $X = Bl_p Y$ and C is the exceptional divisor of this blow-up.*

Proof. None given. □

In practice, we classify **minimal surfaces**, which are those surfaces with no such C (i.e. those X which are not the blow-up of something else).

Definition 7.2. *We say $\varphi : X \to Y$ is* ***meromorphic*** *if $\varphi : X \backslash Z \to Y$ is holomorphic for Z an analytic hypersurface.*

Definition 7.3. *We say X, Y are* ***bimeromorphic*** *if there is a meromorphic $\varphi : X \to Y$ with meromorphic inverse.*

It turns out that all bimeromorphic maps between surfaces are compositions of blow-ups and blow-downs.

Set $P_r := \dim_{\mathbb{C}}\left(H^0(X, K_X^{\otimes r})\right)$, called the **plurigenera**. These are a bimeromorphic invariant. We define the **Kodaira dimension** $K(X)$ to be the growth of P_r in r. So:

- $K(X) = -\infty$ if $P_r = 0$ for all r.
- $K(X) = 0$ if $P_r \in \{0, 1\}$ for all r sufficiently large.
- $K(X) = 1$ if $\exists c$ with $P_r < cr$ for all r sufficiently large.
- $K(X) = 2$ otherwise.

In general we have:

$$K(X) = \limsup_{r\to\infty} \frac{\log\left(\dim_{\mathbb{C}}\left(H^0(X, K_X^{\otimes r})\right)\right)}{\log(r)} = \limsup_{r\to\infty} \frac{\log(P_r)}{\log(r)}.$$

Let us look at each case of $K(X)$.

- If $K(X) = -\infty$: All of these X are projective. The possible classes are:

(i) Rational surface, $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and Σ_n, with $\pi : \Sigma_n \to \mathbb{P}^1$ such that $\pi^{-1}(x) \cong \mathbb{P}^1$ for all $x \in \mathbb{P}^1$. In particular Σ_n has a $C \subset \Sigma_n$, $C \cong \mathbb{P}^1$ with $C \cdot C = -n$ (this is what makes Σ_n distinct from Σ_m for $m \neq n$).

Remark: If K_X^* is ample, then X is called **Fano** ($\to$ del Pezzo surfaces), e.g. $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Bl}_{p_1,\dots,p_8}\mathbb{P}^2$ for arbitrary distinct $p_1, \dots, p_8$.

(ii) Ruled surfaces of genus > 0: these have a map $\pi : X \to S$ with $\pi^{-1}(x) \cong \mathbb{P}^1$ for all x, with S a Riemann surface of genus ≥ 1.

- If $K(X) = 0$: Not all of these are projective. The classes are:

(i) Abelian surfaces (complex tori), $\mathbb{C}^2/\Lambda$. These are projective if and only if the Hodge-Riemann relations hold on Λ. Here $K_X \cong \mathcal{O}_X$, $H^0(X, \mathcal{O}_X) = 1$. These can have no divisors.

(ii) K3 surfaces: $K_X \cong \mathcal{O}_X$. In general $K_X \cong \mathcal{O}_X$ is Calabi-Yau. Sometimes these can be non-projective (they form a 20 dimensional family, and a 19 dimensional subspace is projective).

Example: $V(f) \subset \mathbb{P}^3$ where f has degree 4.

(iii) Enriques surfaces: have $K_X^{\otimes 2} \cong \mathcal{O}_X$, but $K_X \cong \mathcal{O}_X$. These are of the form $Y/\mathbb{Z}_2$ for Y a K3 surface.

- If $K(X) = 1$: There is only one class here, of (some) elliptic surfaces, So $\pi : X \to S$ (S a Riemann surface) such that $\pi^{-1}(x)$ is an elliptic curve for $S\backslash\{p_1, \dots, p_k\}$. The other fibres can be singular (and non-reduced). Here $K_X \cdot K_X = 0$.

  Not all elliptic surfaces have $K(X) = 1$ though, e.g. $\mathbb{P}^1 \times E$, for E an elliptic curve. Some elliptic surfaces can be non-projective as well.

  Aside: For $\pi : X \to B$ and F a general fibre, the Litaka conjecture is that we have $k(X) \geq k(B) + k(F)$.

- If $K(X) = 2$: These are "surfaces of general type". These have $K_X \cdot K_X > 0$, and they are wild and difficult to study/ They do have a nice moduli space (generalising $\mathcal{M}_g$) by Giesecker (the Kollär-Shepherd classification).

  We don't know, e.g. what their topology is (in general).

Now we quickly look at the non-Kähler case.

7.2. **Non-Kähler Surfaces.**

If $b_1 = \dim(H^2(X, \mathbb{R}))$ is even, then X is Kähler. Thus wlog we have b_1 being odd.

- If $K(X) = 1$: Can have non-Kähler elliptic surfaces.
- If $K(X) = 0$: Two classes are:

(i) Primary Kodaira surfaces. We construct these via taking an elliptic curve S and $L \to S$ with $\deg(L) \neq 0$. Write $L^* = L\backslash\{0\text{-section}\}$. Then we have $X = L^*/q^{\mathbb{Z}}$ for $q^{\mathbb{Z}}$ an infinite discrete cyclic subgroup of $\mathbb{C}$.

(ii) Secondary Kodaira surfaces. These are of the form $X_{\text{secondary}} = X_{\text{primary}}/G$, i.e. quotient of a finite group G acting on a primary Kodaira surface X_{primary}.

- If $K(X) = -\infty$: Then $b_1(X) = 1$. Then we have:

(i) If $b_2 = 0$, then have Hopf surfaces and $\mathbb{C}^2\backslash\{0\}/G$ for G a discrete group which acts freely on $\mathbb{C}^2\backslash\{0\}$. In one, these are $(\mathbb{C} \times \mathbb{H})/G$, for $\mathbb{H}$ the upper half-plane and G a solvable discrete group. These have no divisors.

(ii) $b_2 = 1$: these were classified by Nakamura (1984) and A Teleman (2005).

(iii) $b_2 > 1$: still an open question! (as of March 2019). There is a guess however.

For $\dim(X) \geq 3$, we try to reduce to K_X^* or K_X ample, or $K_X \cong \mathcal{O}_X$ (this is the "minimal model program"). For K_X ample the minimal model program is basically complete (Birkar-Cascini-Hacon-McKernan paper).

"Most" 3-folds are not projective. "Most" complex 3-folds are not Kähler. The minimal model program fails for non-Kähler 3-folds (example by Pelham Wilson).

End of Lecture Course