Differential Geometry (Part III)

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These notes are produced entirely from the course I took, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. Please send any corrections to pdtwm2@cam.ac.uk

Contents

1. Smooth Manifolds

Definition 1.1. *A **topological manifold** is a Hausdorff, second countable, topological space X, which is locally homeomorphic to $\mathbb{R}^n$ for some (usually fixed) n.*

Recall: *Second countable* means that there is a countable basis for the topology on X, i.e. $\exists \{U_i\}_{i\in\mathbb{N}}$ open in X such that every open set in X is a union of some of the U_i.

Example 1.1. *$\mathbb{R}^n$ is second countable (and hence topological manifolds generalise $\mathbb{R}^n$).*

A **chart centred at $p \in X$** is a pair (U,φ) with U open in the topology of X, with $p \in U$, and

$$\varphi : (U,p) \xrightarrow{\cong} (B^n, 0)$$

where $B^n = B_1^n(0)$ is the unit ball (this could be any open subset of $\mathbb{R}^n$, but wlog take this).

A choice of chart at p defines **local coordinates** on X, $\{x_1,\dots,x_n\}$, near p, based on the pullback of the usual coordinates on $\mathbb{R}^n$.

Given two charts at $p \in X$, we obtain a **transition function** $\psi\circ\varphi^{-1} : \varphi(U\cap V) \to \psi(U\cap V)$, which is a homeomorphism (as ψ and φ are). Wlog by translation, we can assume $\varphi(p) = 0 = \varphi(p)$. Moreover, as both $\varphi(U\cap V)$, $\psi(U\cap V)$ are open subsets of $\mathbb{R}^n$, we can talk about the derivative of the transition map, which leads us to the notion of a differentiable manifold.

Definition 1.2. *A **smooth** (or **differentiable**) **manifold** is a topological manifold X with an **atlas** of charts $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$, i.e. open sets U_α such that they cover X and such that the transition functions*

$$\varphi_{\alpha\beta} = \varphi_\beta\circ\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha\cap U_\beta) \to \varphi_\beta(U_\alpha\cap U_\beta)$$

are smooth diffeomorphisms for all α,β.

A choice of such an atlas is called a **differentiable structure** on X. We can have different differentiable structures on a manifold.

Any smooth atlas defines a maximal atlas, via the maximum atlas which contains it (this is shown via a Zorn's lemma type argument), in which we include all charts $(U_\alpha,\varphi_\alpha)$ such that the transition maps are smooth (i.e. may give "more resolution", and so more open sets).

Definition 1.3. *If M,N are smooth manifolds, then a map $f : M\to N$ is **smooth at $p\in M$** if for any choice of charts (U,φ) at p and (V,ψ) at $f(p)\in N$, the map*

$$\psi\circ f\circ\varphi^{-1} : \varphi(U)\to\psi(V)$$

is smooth where it is defined.

[Note that since both $\varphi(U), \psi(V) \subset \mathbb{R}^n$ are open, we can talk about the smoothness of this map in the classical sense.]

We say that f is ***smooth*** *if it is smooth at each $p \in M$.*

Note: This is a well-defined notion and does not depend on the choice of charts (this is because the transition maps are smooth - this is an **Exercise** to show).

Definition 1.4. *We say smooth manifolds M, N are* ***diffeomorphic*** *if $\exists$ a smooth bijection $f : M \to N$ with smooth inverse.*

We will tend to write M^n for an n-dimensional smooth manifold.

Example 1.2. *$\mathbb{R}^n$ is a smooth manifold, with an atlas of one chart, $(\mathbb{R}^n, id_{\mathbb{R}^n})$.*

Example 1.3. *S^1 is a smooth manifold. [Recall that $S^1 = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.]*

To see this, we can use stereographic projection to define charts. We have: $\varphi = \varphi_N : S^n \backslash \{N\} \to \mathbb{R}^n$ a homeomorphism (stereographic projection from the north pole, N). We then also have stereographic projection from the south pole as well. These maps $\Longrightarrow S^n$ is a topological manifold (pullback from $\mathbb{R}^n$).

To check it is a smooth manifold, we need to check the transition maps. There is only one to check, and it can be found to be:

$$\underbrace{(x_0, x_1, \ldots, x_{n-1})}_{\in \mathbb{R}^n \cong \{x_n = 0\} \subset \mathbb{R}^{n+1}} \longmapsto \left(\frac{x_0}{|x|^2}, \ldots, \frac{x_{n-1}}{|x|^2} \right)$$

which is smooth as a map on $\mathbb{R}^n \backslash \{0\}$ [All of this is an ***Exercise*** *to check]. Note that this map is also self-inverse. So hence S^n is a smooth manifold.*

Example 1.4. *Let $\mathbb{R}P^n$, the* ***real projective n-space***, *be the set of lines through the origin in $\mathbb{R}^{n+1}$, i.e.*

$$\mathbb{R}P^n := \{v \in \mathbb{R}^{n+1} \backslash \{0\}\} / \sim = S^n / \{\pm 1\}$$

where $v \sim \lambda v \ \forall \lambda \in \mathbb{R} \backslash \{0\}$, and the second equality comes from contracting the lines down onto S^n (as they have been identified), and then just identifying antipodal points.

Denote points in $\mathbb{R}P^n$ by homogeneous coordinates, $[x_0 : \cdots : x_n]$, and so we have

$$[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n] \quad \text{if } \lambda \in \mathbb{R} \backslash \{0\} \text{ and not all } x_i = 0.$$

First, we shall show that this is a topological manifold.

Let $U_i = \{x_i \neq 0\}$. Then set $\varphi_i : U_i \to \mathbb{R}^n$ by:

$$\varphi_i([x_0 : \cdots : x_n]) = \frac{1}{x_i}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

(i.e. omit x_i and divide by it). Then this is bijective, and in fact is a homeomorphism when $\mathbb{R}P^n$ has the quotient topology from $\mathbb{R}^{n+1}\backslash\{0\}$. This shows that $\mathbb{R}P^n$ is a topological manifold.

*To see that it is a smooth manifold, we need to check the transition functions. One can compute [**Exercise** to check]*

$$\varphi_j \circ \varphi_i^{-1} : (y_1, \ldots, y_n) \longmapsto \frac{1}{y_j}(y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_{j-1}, y_{j+1}, \ldots y_n)$$

i.e. omit y_j and insert a 1 in the i'th place. Again this is a smooth map on its domain of definition, $\{y_j \neq 0\}$. So hence $\mathbb{R}P^n$ is a smooth manifold for all n.

Let M be an n-manifold and N a k-manifold. Let $f : M \to N$ be smooth. Then given a chart (U, φ) at p and a chart (V, ψ) at $f(p)$, we get a map $\psi \circ f \circ \varphi^{-1}$ defined on an open set in $\mathbb{R}^n$, valued in an open set in $\mathbb{R}^k$. So let $D(\psi \circ f \varphi^{-1})\big|_{\varphi(p)}$ be the corresponding Jacobian matrix of partial derivatives.

Then the **rank** of the matrix $D(\psi \circ f \circ \varphi^{-1})\big|_{\varphi(p)}$ is independent of the choice of charts φ, ψ (essentially by the chain rule) [**Exercise** to check].

Definition 1.5. *We say that f is:*

- *An **immersion** at p if $D(\psi \circ f \circ \varphi^{-1})\big|_{\varphi(p)}$ is injective.*
- *A **submersion** at p if $D(\psi \circ f \circ \varphi^{-1})\big|_{\varphi(p)}$ is surjective.*

Definition 1.6. *If $i : N^k \hookrightarrow M^n$ is a smooth map of manifolds which is injective, then we say that $i(N)$ is a **submanifold** of M if:*

- *i is an immersion*
- *i is a homeomorphism onto its image*

i.e. we want the given manifold topology on N to agree with the subspace topology on $i(N)$ inherited from the topology on M, i.e. we want the topologies to be compatible (we wouldn't want different differentiable structures on the same smooth manifold to be compatible!).

Contrast this with the following: Consider the torus $T^2 = S^1 \times S^1 \simeq \mathbb{R}/\mathbb{Z}$, which is a manifold [**Exercise** to check - or just note that the projection of a manifold is a manifold]. Then the image of $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ inside T^2 (via the projection map) is a submanifold (i.e. one of the inner circles).

Alternatively, the map $i : \mathbb{R} \to T^2$, sending $t \mapsto [t, \alpha t]$ (the equivalence class), where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is irrational, is injective and an immersion everywhere, but the subspace topology on $i(\mathbb{R})$ is not the given topology on $\mathbb{R}$ (because α is irrational, so there is no periodicity in the curve).

Definition 1.7. *Let $f : M^n \to N^k$ be smooth. Then we say that $q \in N$ is a* ***regular value*** *of f if $\forall p \in f^{-1}(q)$, we have that f is a submersion at p (such p are called* ***regular points****),*

i.e. for all charts (U, φ) at p and charts (V, ψ) at q, the corresponding Jacobian has rank k (i.e. maximal rank).

Theorem 1.1 (The Pre-Image Theorem)**.** *Suppose $f : M^n \to N^k$ is a smooth map, and $q \in N$ is a regular value of f. Then, $f^{-1}(q) \hookrightarrow M$ (inclusion) is either empty, or an $(n-k)$-dimensional submanifold of M. In particular, it is a manifold.*

Proof (Sketch). Let $Y = f^{-1}(q)$. Then Y being second countable and Hausdorff is inherited directly from M^n.

Now let us pick $p \in f^{-1}(q)$, and choose local coordinates (via charts), $x_1, \ldots, x_n$ at p and $y_1, \ldots, y_k$ at q. Then in these coordinates, f takes the form: $f : U \to V$, with wlog $f(0) = 0$ and $f = (f_1 \ldots, f_k)$ (here, $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ are both open).

Now extend f to a map $F : \mathbb{R}^n \to \mathbb{R}^n$ (as $k \leq n$), sending

$$F(x_1, \ldots, x_n) = (f_1(x), \ldots, f_k(x), x_{k+1}, \ldots, x_n),$$

which is a map defined on open neighbourhoods of $0 \in \mathbb{R}^n$.

Then since q is a regular value, $\mathrm{D}f|_p$ is surjective, and hence by reordering the coordinates as necessary (so the first k will span), we can assume wlog that the $k \times k$ matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq k}$ is surjective, and hence is invertible (as this is equivalent for linear maps, i.e. need the first k coordinates to map to a space of dimension k).

So hence we have

$$\mathrm{D}F|_0 = \begin{bmatrix} \left(\frac{\partial f_i}{\partial x_j}\right)_{k \times k} & 0 \\ \left(\frac{\partial f_i}{\partial x_j}\right)_{(n-k) \times k} & I \end{bmatrix}$$

where I is an identity matrix. So hence we see that $\mathrm{D}F|_0$ is an isomorphism, and so by the inverse function theorem, we know that F is locally invertible, i.e. $(f_1, \ldots, f_k, x_{k+1}, \ldots, x_n)$ form a local system of coordinates at $p \in M^n$.

In these coordinates, the projection to $(x_{k+1}, \ldots, x_n)$ defines a chart on Y near p (as then the f is constant on this chart). Then by inspection, this shows that Y is a manifold such that the inclusion $Y \hookrightarrow M^n$ is smooth (**check**), and $\exists$ a local smooth projection $M \to Y$ (via removing the f_i) onto a neighbourhood of p [**Exercise** to check that the required transition maps are smooth].

$\square$

Example 1.5. *If $f : M^n \to N^n$ is smooth and M, N are compact of the same dimension (here, compact without boundary $\equiv$ closed), then if $q \in N$ is regular, we see that $f^{-1}(q)$ is finite.*

Exercise: *Show that $\left|f^{-1}(q)\right|$ is a locally constant function on the set of regular values q.*

Example 1.6. *Let $O(n) = \{A \in Mat_n(\mathbb{R}) \cong \mathbb{R}^{n^2} : A^T A = I_n\}$ be the orthogonal group. Then consider:*

$$f : Mat_n(\mathbb{R}) \to Sym_n(\mathbb{R})$$

where $Sym_n(\mathbb{R}) = \{\text{Symmetric } n \times n \text{ matrices}\}$, with $f(A) = A^T A$. Then one can check that

$$Df|_A(H) = H^T A + A^T H$$

*is this linear map, in the usual multi-variable differentiation sense. So, $Df|_I(H) = H + H^T$, which is certainly surjective, and so (**Exercise** to show) we find that I is a regular value of f. So as $f^{-1}(I) = O(n)$, we see that $O(n)$ is a smooth (sub)manifold.*

Remark: *Sard's Theorem* says that if $f : M^n \to N^k$ is a smooth map, then the regular values of f are dense in N^k.

1.1. **Tangent Vectors and Tangent Bundles.**

Let $\Sigma \subset \mathbb{R}^N$ be a smooth submanifold of dimension k (i.e. embed our manifold Σ in some $\mathbb{R}^N$.).

Then if $p \in \Sigma$, and $\gamma : (-\varepsilon, \varepsilon) \to \Sigma \subset \mathbb{R}^N$ is a smooth curve through p, with $\gamma(0) = p$, then $\gamma'(0)$ is a vector in $\mathbb{R}^N$ (which is based at p, not 0). The set of all such vectors as one varies γ (for fixed p) forms a k-dimensional affine subspace of $\mathbb{R}^N$ passing through p (this is because if we take a chart about $p \in \Sigma$, then Σ locally looks like $\mathbb{R}^k$ about p, and so we get a tangent for each direction).

Formally, if we have a chart $\varphi : (U, p) \to (B^k, 0)$ at p for Σ, and if $i : \Sigma \hookrightarrow \mathbb{R}^N$ is the embedding, then we can define:

$$T_p\Sigma = \text{Image}\left(\mathrm{D}(i \circ \varphi^{-1})\Big|_{\varphi(p)}\right)$$

which is called the **tangent space to Σ at p**. It is the space of all such tangents. Certainly via φ, any γ on Σ gives a curve $\tilde{\gamma}$ through $0 \in B^k$, whose tangent vector is in the domain of this map.

Globally on Σ, we can then define the **tangent bundle** to be

$$T\Sigma := \{(p, v) \in \mathbb{R}^N \times \mathbb{R}^N : p \in \Sigma,\ v \in T_x\Sigma\} \equiv \coprod_{p \in \Sigma} T_p\Sigma$$

is the set of all such tangent vectors, indexed by the point on Σ.

Note: This provisional definition of $T_p\Sigma$ does not depend on the choice of chart φ, since if (V, ψ) were another chart, then:

$$i \circ \psi^{-1} = \left(i \circ \varphi^{-1}\right) \circ \underbrace{\left(\varphi \circ \psi^{-1}\right)}_{\text{derivative is an isomorphism}}$$

where both these maps are defined. So the derivatives have the same image.

This is a provisional definition of the tangent space because it relies on us embedding our manifold in some overall $\mathbb{R}^N$, which isn't necessarily always the case (or at least at the moment is not clear if always is the case). We will soon see other constructions, which are less geometrically intuitive, but are more useful in other ways.

The space $T\Sigma$, topologised as a subspace of $\mathbb{R}^N \times \mathbb{R}^N$, defined as above, has the following features:

(i) $\exists$ a canonical projection $\pi : T\Sigma \to \Sigma$ via $\pi(p,v) = p$, whose **fibre** $\pi^{-1}(p)$ at $p \in \Sigma$ is naturally a k-dimensional vector space (as this is simply T_pM).

(ii) The map $\pi : T\Sigma \to \Sigma$ is **locally trivial** in the sense that, given a chart (U,φ) at $p \in \Sigma$, we can identify:
$$T\Sigma|_U := \pi^{-1}(U) = \coprod_{q\in U} T_q\Sigma \cong U \times \mathbb{R}^k,$$
where the isomorphism is simply via
$$(q,v) \longmapsto (\varphi(q), v)$$
$$\mathrm{D}(i\circ\varphi^{-1})\big|_{\varphi(q)}(v) \longleftarrow (q,v)$$
where we have used that each tangent space is essentially just a copy of $\mathbb{R}^k$.

These observations motivate the following generalisation:

Definition 1.8. *If M is a smooth manifold, then a **smooth vector bundle** $\pi : E \to M$ of **rank k** comprises of:*

(i) A smooth (n+k)-manifold E

(ii) A <u>submersion</u> $\pi : E \to M$ such that each fibre is $\cong \mathbb{R}^k$, a k-dimensional vector space

*(iii) π is **locally trivial**, in the sense that, $\forall p \in M$, $\exists$ an open neighbourhood U of p and a map Φ such that the diagram*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi\ \cong"] & U \times \mathbb{R}^k \\ \pi^{-1}(q) \arrow[u, hook, "\text{inclusion}"] \arrow[r, "\cong"] & \{q\} \times \mathbb{R}^k \arrow[u, hook, "\text{inclusion}"'] \end{tikzcd}$$

commutes for all $q \in U$, with the map on the base being a linear isomorphism.

Remark: In some sense, we are associating a vector space to each point of M. But when neighbourhoods overlap, we need some kind of compatibility condition between the vector spaces. This is what the local trivialisation condition is doing, and the need for the base map being a linear isomorphism.

Example 1.7. *$T\Sigma$ is a smooth vector bundle of rank n (so in some sense, a smooth vector bundle can be thought of as all 'tangent vectors' to M, where 'tangent vectors' is generalised).*

So note how all of the above depended on embedding Σ in some $\mathbb{R}^N$, Our aim is to give an intrinsic definition on the tangent bundle, TM, of a smooth manifold M^n (i.e. one which only depends on

M itself, and not how it embeds in some larger space), which is a rank n vector bundle as defined before. We will want the different notions to be isomorphic, but not using an embedding of M in some $\mathbb{R}^N$. The reason for this is that it is not clear in the above construction whether TM depends on the embedding we choose - i.e. does rotation change TM?

The key thing is to decide how to define T_pM, the tangent space of M at p, i.e. how to define a tangent vector at a point p of M. There are 3 definitions; the first is intuitive geometrically (curves + germs). The second is clean and provides us with a nice basis of each tangent space, but not intuitive (derivations). The third is ugly, but computable and allows us to easily define more general bundles (cocycles).

1.1.1. *Definition via Curves.*

Definition 1.9. *A* ***germ of a curve*** *on M at p is a map $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, up to the equivalence relation $\sim$, where $\gamma \sim \tau$ if $\exists \delta < \min(\varepsilon_\gamma, \varepsilon_\tau)$ such that $\gamma \equiv \tau$ when restricted to $(-\delta, \delta)$.*

This definition is just saying that we only care about the behaviour of the curve about p, since this will determine the tangent.

Then we define T_pM to be the **equivalence classes of germs of curves at p**, under the second equivalence relation (which is on germs, which are already themselves equivalence classes)

$$\gamma_1 \sim \gamma_2 \iff \forall \text{ charts } (U, \varphi) \text{ at } p, \text{ if } \varphi : U \to B^n \subset \mathbb{R}^n \text{ with } p \mapsto 0, \text{ then we have}$$
$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$$

i.e. if the tangent vectors at the origin at the same.

Clearly this only depends on the germ (and not the curve), since they agree on some neighbourhood about the origin, and so they give the same tangent vector there. Although note that different germs can have the same tangent vector (i.e. think of a straight line and a curve touching the line).

Then we define: $[\gamma_1] + [\gamma_2] = [\delta]$, where δ is such that in all charts (U, φ), we have

$$(\varphi \circ \delta)'(0) = (\varphi \circ \delta_1)'(0) + (\varphi \circ \delta_2)'(0).$$

[Think this through why this makes sense, i.e. why it is independent of the choice of representative of germ and of the chart.]

Note: We see that $T_pM \cong \mathbb{R}^n$, where $n = \dim(M)$, since if $\varphi : (U, p) \to (B^n, 0)$ is a chart, then

$$\alpha : TM|_U = \bigcup_{q \in U} T_qM \xrightarrow{\cong} U \times \mathbb{R}^n, \quad \text{where} \quad \alpha(q, [\delta]) = (q, (\varphi \circ \gamma)'(0))$$

is an isomorphism. It is easy to see that α is a bijection (i.e. fix q, then vary γ to get all of $\mathbb{R}^n$). Then we can define a topology on $TM|_U$ by declaring that this α is a homeomorphism, i.e. the open sets in $TM|_U$ are the pullbacks via α of open sets in $U \times \mathbb{R}^n$.

Exercise: Show that the chart overlaps do indeed show that TM is a smooth vector bundle [It is important to check/think about this.]

1.1.2. *Definition via Derivations.*

Let $C^\infty(M) = \{\text{Smooth functions } M \to \mathbb{R}\}$, where M is a smooth manifold. Then, $C^\infty(M)$ is an infinite dimensional vector space.

Definition 1.10. *A* ***derivation*** *at $p \in M$ is a linear map $\alpha_p : C^\infty(M) \to \mathbb{R}$ such that*

$$\alpha_p(f \cdot g) = f(p) \cdot \alpha_p(g) + \alpha_p(f) \cdot g(p)$$

i.e. like the product rule for derivatives at a point.

Observe that this product relation for a derivation is linear in α_p, and so [**Exercise** to check] the set of derivations at p is naturally a vector space.

Lemma 1.1 (Bump Functions Exist)**.** *If $p \in M$, and $p \in U \subset M$ is open, then $\exists V \subsetneq U$, $p \in V$ with V open in M and a $f \in C^\infty(M, \mathbb{R})$ such that*

$$f = \begin{cases} 1 & \text{on } V \\ 0 & \text{on } M \backslash U \end{cases}$$

i.e. f is one about p in U, but vanishes outside U. Such an f is called a ***bump function****.*

Proof sketch. It suffices to check this locally on a neighbourhood of $0 \in \mathbb{R}^n$, by considering charts, etc.

So consider $\alpha : \mathbb{R} \to \mathbb{R}$ given by:

$$\alpha(t) = \begin{cases} e^{-\frac{1}{t^2}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then let $\beta : \mathbb{R} \to \mathbb{R}$ be:

$$\beta(t) = \frac{\alpha(t)}{\alpha(t) + \alpha(1-t)}$$

which is 0 for $t \leq 0$, and 1 for $t \geq 1$. Then define $\gamma : \mathbb{R} \to \mathbb{R}$ be

$$\gamma(t) = \beta(2+t)\beta(2-t).$$

This gives the result for $\mathbb{R}$. Then if we set $f : \mathbb{R}^n \to \mathbb{R}$ to be

$$f(x) = \gamma(x_1) \cdots \gamma(x_n)$$

then we get the result for $V =$ unit cube, and $U =$ cube of side length 2 in $\mathbb{R}^n$. Then by scaling and translation we get the general result.

□

Note: Such bump functions do not exist over $\mathbb{C}$, by Liouville's theorem. This is what gives differential geometry its different style to the complex manifolds course next term: so anything we do with these functions (which will be a lot) we can't do for the complex manifolds case (where we require transition maps to be biholomorphic, etc.)

Now we prove a local characterisation of derivations:

Lemma 1.2. *Let l be a derivation at $p \in M$. Then if $f, g \in C^\infty(M)$ and $f \equiv g$ in some neighbourhood of p, then $l(f) = l(g)$.*

Proof. Let $h = f - g$ (note that we can subtract as f, g are $\mathbb{R}$-valued). Then by the previous lemma, if $p \in V \subset h^{-1}(0)$ is open, we know that $\exists$ a bump function B such that $B(p) = 1$, and

$$\text{supp}(B) := \overline{\{x \in M : h(x) = 0\}} \subset V.$$

So hence,

$$0 = l(0) = l(\underbrace{h \cdot B}_{\text{identically zero}}) = \underbrace{h(p)}_{=0} \cdot l(B) + l(h) \cdot \underbrace{B(p)}_{=1} = l(h).$$

So hence as $l(h) = l(f) - l(g)$ by linearity of l, we are done.

□

So in fact we could define derivations at p as linear maps on $C_p^\infty(M)$, which is the vector space of germs of smooth functions at p (such that they satisfy the product condition, etc), since this lemma shows that l is independent of the choice of function in the germ.

Lemma 1.3. *The vector space Der_p of derivations at p has dimension n (= $\dim(M)$).*

Proof. It again suffices to prove this for $0 \in \mathbb{R}^n$ by working locally in a chart.

Note that the operators $\left.\frac{\partial}{\partial x_i}\right|_0$, for $1 \leq i \leq n$, do define derivations are 0 [**Exercise** to check]. These are also linearly independent as derivations [**Exercise** to check], and so we just need to check that they span Der_p (as there are n of them).

So let $f \in C^\infty(\mathbb{R}^n)$, and suppose $f(0) = 0$. Suppose f is supported on a convex open neighbourhood U of $\{0\}$. Fix $l \in \text{Der}_p$. Then we need to show that the action of l on f is the same as some action of the $\left.\frac{\partial}{\partial x_i}\right|_0$.

Note that $l(\text{constant function}) = 0$, since $l(1) = l(1 \cdot 1) = 2l(1) \Rightarrow l(1) = 0$, where we have used the product property of derivations.

Key Observation: $\exists$ functions $g_i(x)$ such that $g_i(0) = \left.\frac{\partial f}{\partial x_i}\right|_0$, and such that $f(x) = \sum_{i=1}^n x_i g_i(x)$. This will imply that $l \in \text{Span}\langle \left.\frac{\partial}{\partial x_1}\right|_0, \ldots, \left.\frac{\partial}{\partial x_n}\right|_0 \rangle$.

To find the g_i, note that if $h_x(t) = f(tx)$, then:

$$f(x) = \int_0^1 h_x'(t) \, \mathrm{d}t = \sum_i \int_0^1 x_i \cdot \frac{\partial f}{\partial x_i}(tx) \, \mathrm{d}t,$$

by the chain rule. So set

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,\mathrm{d}t.$$

So hence $f(x) = \sum_{i=1}^n x_i g_i(x)$. So we have:

$$l(f) = l(\sum_i x_i g_i) = \sum_i l(x_i g_i) = \sum_i l(x_i) g_i(0) = \sum_i a_i \left.\frac{\partial f}{\partial x_i}\right|_0 = \left(\sum_i a_i \left.\frac{\partial}{\partial x_i}\right|_0\right)(f),$$

where $a_i = l(x_i) \in \mathbb{R}$ are some constants, independent of f. So hence as f was arbitrary, we have

$$l = \sum_i a_i \left.\frac{\partial}{\partial x_i}\right|_0$$

and so we are done.

□

Definition 1.11. *Now we define T_pM, the **tangent space** of M at p, to be Der_p, the vector space of derivations at p.*

This definition is nice because we instantly have a basis of each tangent space via the derivatives. So now we need to show that this definition via derivations coincides with the other.

Remark: Note that we can view these as maps on $C^\infty(M)$ or on $C_p^\infty(M)$. Also, $C_p^\infty(M)$ has a well-defined subspace of germs of functions with vanishing derivative at p, which we denote by K_p. So some authors define:

$$T_p(M) := \left(C_p^\infty(M)/K_p\right)^*$$

i.e. throw away things with zero derivative at p.

The useful thing about this definition of the tangent space is that it has a clear vector space structure, via the partial derivatives being the basis vectors. So explicitly, if $U \subset M$ is a chart a p with defining local coordinates $\{x_1, \dots, x_n\}$ at p, then

$$T_p(M) = \mathrm{Span}\langle \left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p \rangle$$

and if $f \in C^\infty(M)$, then locally f is $f(x_1, \dots, x_n)$, and via the above calculation,

$$\underbrace{\left.\frac{\partial}{\partial x_i}\right|_p}_{\text{derivation = derivative in this direction}}(f) = \left.\frac{\partial f}{\partial x_i}\right|_p$$

using the local coordinate representation.

So now, if $\{y_1, \dots, y_n\}$ are different local coordinates at p, then we also have $T_p(M) = \mathrm{Span}\langle \left.\frac{\partial}{\partial y_1}\right|_p, \dots, \left.\frac{\partial}{\partial y_n}\right|_p \rangle$. Then by the above, we can express the basis vectors in terms of each other, and so we find

$$(\star) \qquad \left.\frac{\partial}{\partial x_i}\right|_p = \sum_{j=1}^n \left.\frac{\partial y_j}{\partial x_i}\right|_p \cdot \left.\frac{\partial}{\partial y_j}\right|_p$$

i.e. the usual chain rule, where we are differentiating the functions y_j viewed as functions of only the x_i.

Remarks: Note the following.

(i) When we defined T_pM via germs of curves, it had an obvious functoriality property, that being:
"If $f : M \to N$ is a smooth map, then we get a map $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ via
$$[p,\gamma] \longmapsto [f(p), f \circ \gamma]$$
for a curve/germ γ at p."
Similarly, we get a similar property in the derivation definition:
"If $f : M \to N$ is a smooth map, and $l : C^\infty(M) \to \mathbb{R}$ is a derivation, then we get a derivation $\tilde{l} : C^\infty(N) \to \mathbb{R}$ via: $\tilde{l}(g) = l(g \circ f)$."

(ii) If $\gamma : (-\varepsilon, \varepsilon) \to M$ sends $0 \longmapsto p$ is a germ of a curve at p, then we obtain a derivation $C^\infty(M) \to \mathbb{R}$ via: $f \longmapsto (f \circ \gamma)'(0)$.
So hence a tangent vector in the sense of germs of curves yields one in the sense of derivations.

Hence we see that these two definitions/notions are equivalent.

1.1.3. *Definition via Cocycles.*

Recall that a smooth vector bundle $E \to M$ of rank k is by definition locally trivial, i.e. $\forall p \in M$, $\exists U \ni p$ open such that $\forall q$, the following diagram commutes:

$$\begin{tikzcd} E|_U \arrow[r, "\cong"] & U \times \mathbb{R}^k \\ E_q \arrow[u, hook, "\text{inclusion}"] \arrow[r, "\cong"] & \{q\} \times \mathbb{R}^k \arrow[u, hook, "\text{inclusion}"'] \end{tikzcd}$$

where $E_q = E|_q$. So if we have two trivialising neighbourhoods U, V of p, then:
$$E|_U \cong U \times \mathbb{R}^k \supset (U \cap V) \times \mathbb{R}^k$$
$$E|_V \cong V \times \mathbb{R}^k \supset (U \cap V) \times \mathbb{R}^k.$$
So over $U \cap V$, the trivialisations φ_U and φ_V differ by a map:
$$\varphi_{UV} : U \cap V \to \mathrm{GL}_k(\mathbb{R})$$
where $q \in U \cap V$ is sent to the composite map
$$\{q\} \times \mathbb{R}^k \xhookrightarrow{\varphi_U} E|_{U\cap V} \xrightarrow{\varphi_V} E|_{U\cap V} \longrightarrow \{q\} \times \mathbb{R}^k$$
(noting that $\{q\}\times\mathbb{R}^k \cong \mathbb{R}^k$, i.e. $\varphi_{UV}(q)$ is the map $\{q\}\times\mathbb{R}^k \to E_q \to \{q\}\times\mathbb{R}^k$). So suppose now that we have $M = \{U_\alpha\}_{\alpha\in A}$ is an atlas of trivialising neighbourhoods. Then the maps $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$ satisfy:

- $\psi_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$
- $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} \circ \psi_{\gamma\alpha} = \mathrm{id}$ (as each inverse appears in the chain).

This pair of conditions is called the **cocycle condition**.

The point is that these conditions are actually all we need to build a vector bundle from scratch: indeed, given M and an open cover $\{U_\alpha\}_{\alpha\in A}$, and maps $\psi_{\alpha\beta}$ satisfying the cocycle condition, then we can build a vector bundle $E \to M$ by taking

$$\coprod_{\alpha\in A}\left(U_\alpha \times \mathbb{R}^k\right)$$

and gluing together/imposing the equivalence relation:

$$\underbrace{(q,v)}_{\in U_\alpha\times\mathbb{R}^k} \sim \underbrace{(q,\psi_{\alpha\beta}(v))}_{\in U_\beta\times\mathbb{R}^k},$$

for $q \in U_\alpha \cap U_\beta$, i.e. identify all of the 'same' points when gluing the same parts together.

Exercise: Show that the resulting space yields a well-defined vector bundle.

Now, given a smooth manifold M and a collection/atlas of charts $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$, so that

$$M = \bigcup_{\alpha\in A} U_\alpha, \quad \text{and} \quad \varphi_\alpha : U_\alpha \to B^n \subset \mathbb{R}^n,$$

we can declare $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ to be:

$$p \longmapsto \mathrm{D}\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)\Big|_{\varphi_\alpha(p)}$$

where D represents the Jacobian of the map. Then clearly $\varphi_{\alpha\alpha} = \mathrm{id}$, and the chain rule implies the rest of the cocycle condition. Then because we know that the cocycle condition determines the tangent bundle, this then shows that this **is** the tangent bundle TM of M, i.e. this construction agrees with the other two.

[Compare this with ($\star$) $\to$ the coordinate transform above comes from the Jacobian.]

Remark: The point of cocyles is that they allow you to show more complicated bundles are actually bundles. If you can construct the cocycles of, say, the dual bundle or pullback bundle from those of the original bundle, you can show that the new bundle is actually a bundle. This tends to be much easier than other methods.

So now we have gone through the three equivalent constructions of the tangent space. The second is the most useful, as it gives us a basis to work with.

The following meta-theorem may be psychologically reassuring:

Theorem 1.2. *Let $F : M \longmapsto (TM)^\#$ be an association of a smooth vector bundle of rank n to each smooth n-dimensional manifold M, such that:*

(i) If $f : M \to N$ is smooth, then there is an induced smooth morphism of bundles $f_\# : (TM)^\# \to (TN)^\#$ such that

$$(id_M)_\# = id_{(TM)^\#} \quad \text{and} \quad (f\circ g)_\# = f_\# \circ g_\#.$$

(ii) $(T\mathbb{R}^n)^\# = \mathbb{R}^n \times \mathbb{R}^n$ *in such a way that if* $f : \mathbb{R}^n \to \mathbb{R}^k$ *is smooth, then the diagram*

$$\begin{array}{ccc} (T\mathbb{R}^n)^\# & \xrightarrow{f_\#} & (T\mathbb{R}^k)^\# \\ {\scriptstyle identity}\updownarrow & & \updownarrow{\scriptstyle identity} \\ \mathbb{R}^n \times \mathbb{R}^n & \xrightarrow[(f,Df)]{} & \mathbb{R}^k \times \mathbb{R}^k \end{array}$$

commutes.

(iii) *If* $U \subset M$ *is open, then* $(TU)^\# = (TM)^\#\big|_U$*, and if* $f : M \to N$ *is smooth, then*

$$(f|_U)_\# = (f_\#)|_{(TU)^\#}.$$

Then if we have all of these, then we have

$$(TM)^\# \xrightarrow{\cong} TM$$

where TM *is any formulation of* TM.

Proof. None given (See Chapter 3 of Spivak, Vol 1: "A comprehensive introduction to differential geometry"). □

Note however that all this theorem is saying is that the only smooth vector bundle with these properties is the tangent bundle, which we constructed before.

Definition 1.12. *A **(smooth) section** of a (smooth) vector bundle* $\pi : E \to M$ *is a map* $s : M \to E$ *such that* $\pi \circ s = id_M$*, i.e.* $s(p) \in E_p$ *for all* $p \in M$.

*The **vector space of sections of*** $\boldsymbol{E}$ *is denoted* $\boldsymbol{\Gamma(E)}$ *(i.e. we can add sections pointwise, as their images lie in the same linear space, i.e. the tangent space).*

A section of the tangent bundle (i.e. the special case when $E = TM$*) is called a **vector field**.*

A section/vector field can be imagined by choosing a tangent vector at each point of M.

You can imagine taking a point $p \in M$, and then pushing p on M, following the vector field around M. This is would be a **flow** on M. Hence a vector field can be integrated to get a **flow on** $\boldsymbol{M}$, i.e. a family of smooth maps $\varphi_t : M \to M$, where $\varphi_t(p)$ tells you the new position of p after time t of flowing. The following proposition gives us a relation between tangent vectors and flows.

Proposition 1.1. *Let* $X \in \Gamma(TM)$ *be a section/vector field. Then,* $\forall p \in M$, $\exists \varepsilon > 0$ *and an open neighbourhood* $U_p \ni p$ *of* p *in* M*, and a unique family of smooth maps* $\varphi_t : U_p \to M$*, called a **flow**, such that:*

(i) φ_t *is defined for* $|t| < \varepsilon$, *and the resulting map* $\varphi : (-\varepsilon, \varepsilon) \times U_p \to M$ *is smooth, where* $\varphi(t, p) = \varphi_t(p)$.

(ii) *If* $|t|, |s|, |t+s| < \varepsilon$, *and* $x, \varphi_t(x) \in U_p$, *then*

$$\varphi_{t+s}(x) = (\varphi_s \circ \varphi_t)(x).$$

So in particular, $\varphi_0 = id$.

(iii) *If* $f \in C_p^\infty(M)$, *then as* $X(p)$ *is a derivation, we have, the value of* $(X(p))(f) \equiv (X \cdot f)|_p$, *is:*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \varphi_t)(p).$$

Comment: The slogan of this proposition is:

"Tangent vectors are derivations infinitesimally, whilst vector fields are derivations globally."

So we see that given $X \in \Gamma(TM)$ a vector field, and $f \in C^\infty(M)$, then we get a new function $X \cdot f$ on M. Then by (iii), this function is smooth. So hence X gives a map: $C^\infty(M) \to C^\infty(M)$ via: $f \longmapsto X \cdot f$.

Then noting that $X \cdot (fg) = (X \cdot f)g + f(X \cdot g)$, we see that [**Exercise** to check] that the map $f \longmapsto X \cdot f$ gives derivations of $C^\infty(M)$ (via evaluation at the appropriate point).

Proof Sketch. This proposition is really just a version of the existence and uniqueness of solutions to ODE's. We will define φ_t via a solution to property (iii), which will have the other properties ((ii) is then a consequence of uniqueness of solutions, and (i) a consequence of local smoothness).

Write $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ in local coordinates, where $a_i \in C^\infty(U)$ for some chart U. Then, (iii) says/requires:

$$X(f) = \sum_{i=1}^n a_i \frac{\mathrm{d}f}{\mathrm{d}x_i} = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_t) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \frac{\mathrm{d}(\varphi_t)_i}{\mathrm{d}t},$$

by the chain rule. So this is really just asking us to solve:

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = a_i(\varphi_1, \ldots, \varphi_n), \quad \text{where} \quad \varphi_i = \varphi_i(x_1, \ldots, x_n, t), \quad \forall i,$$

subject to $\varphi_i(x_1, \ldots, x_n, 0) = x_i \ \forall i$, i.e. $\varphi_0 = \mathrm{id}$.

So locally, Picard's theorem (i.e. Picard-Lindelöf) - noting that the a_i are smooth and so are locally Lipschitz - gives us unique smooth solutions locally, and so essentially proves the theorem.

□

Remark: An alternative way of looking at this proposition is the existence of **integral curves**, i.e. a curve γ such that $X_{\gamma(t)} = \gamma'(t)$ (intuitively this just says that the tangent of γ at a point is the vector field at that point). Once we have existence of such integral curves (which is just from ODE theory), if we write $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, so that $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$ (in the basis of T_pM, $p = \gamma(0)$, i.e.)

$$\gamma'(t) = \sum_i \gamma_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}$$

then:

$$\begin{aligned} X_p \cdot f = X_{\gamma(0)} &= \sum_i \gamma_i'(0) \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(0)} \\ &= \sum_i \gamma_i'(0) \frac{\partial f}{\partial x_i}(\gamma(0)) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \gamma)(t) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \varphi_t)(p) \end{aligned}$$

where $\varphi_t(p) := \gamma(t)$, where γ is the integral curve through p. Property (ii) of the proposition then comes from uniqueness of integral curves through a point, and property (iii) is simply the above. [Arguably integral curves are the more intuitive object to work with. See Lee's book, page 208, for more details.]

Definition 1.13. *An **integral curve** of X is a path γ such that $\dot{\gamma}(t) = X_{\gamma(t)}$.*

Example 1.8. *If $M = B(1) \subset \mathbb{R}^n$ is the open unit disc, and $X = \frac{\partial}{\partial x}$ is the derivation/constant vector field, then the flow is the translation: $\varphi_t(a, b) = (a + t, b)$ (intuitive, as flow lines are simply parallel to x-direction - also easily calculated from (iii) above). Note that this is only locally defined for some t, or else otherwise the flow would leave $M = B(1)$.*

Lemma 1.4. *Let X be a derivation. Suppose X has compact support (in the usual sense, e.g. this is true if M, the domain of X, if compact itself). Then the flow φ_t of X defines a 1-parameter subgroup $\varphi : \mathbb{R} \to Diff(M)$ of the diffeomorphism group of M.*

Proof. Cover supp(X) by a finite collection of local open neighbourhoods U_{p_i}, which are given as in Proposition 1.1 (we can do this as X has compact support).

Then let $\varepsilon = \min_i \varepsilon_i > 0$. Then since the flows are unique, they must agree on overlaps. Hence as the U_i cover supp(X), we get a well-defined map

$$\varphi : (-\varepsilon, \varepsilon) \times M \to M$$

which is defined by $\varphi(t, p) = (\varphi_i)_t(p)$ if $p \in U_{p_i}$ and zero otherwise (as X is zero outside the union of the U_{p_i}). So now we must just define φ on $\mathbb{R} \times M$, i.e. extend the time parameter. But then we can do this by the $\varphi_{t+s} = \varphi_t \circ \varphi_s$ property: indeed, if $|t| > \varepsilon$, then we can write $t = \frac{\varepsilon}{2} \cdot k + r$, where

$|r| < \frac{\varepsilon}{2}$ and $k \in \mathbb{Z}$. Then let

$$\varphi(t) = \begin{cases} \varphi_{\varepsilon/2} \circ \cdots \circ \varphi_{\varepsilon/2} \circ \varphi_r & \text{if } k > 0 \\ \underbrace{\varphi_{-\varepsilon/2} \circ \cdots \circ \varphi_{-\varepsilon/2}}_{k \text{ times}} \circ \varphi_r & \text{if } k < 0. \end{cases}$$

Then since $\varphi_t \circ \varphi_s = \varphi_{t+s}$, this is well-defined and extends φ to a map $\mathbb{R} \times M \to M$. Then as $\varphi_t^{-1} = \varphi_{-t}$, these maps are diffeomorphisms (as their inverses are smooth), and so we are done.

□

Definition 1.14. *A vector field X is* ***complete*** *if it defines a flow for all time, and hence defines a 1-parameter subgroup $\mathbb{R} \to \mathrm{Diff}(M)$.*

So the above Lemma simply says that, a vector field with compact support is automatically complete.

Lemma 1.5. *If M is a connected n-manifold, then the group $\mathrm{Diff}(M)$ acts transitively on M.*

Proof. Fix $p \in M$, and consider:

$$U = \{q \in M : \exists \psi \in \mathrm{Diff}(M) \text{ such that } \psi(p) = q\}$$
$$U' = \{q \in M : \nexists \psi \in \mathrm{Diff}(M) \text{ such that } \psi(p) = q\}.$$

Then it suffices to prove that both U and U' are open, as M is connected and $U' = M \backslash U$. Note that $U \neq \emptyset$ as $p \in U$, as $\mathrm{id}_M(p) = p$ and $\mathrm{id}_M \in \mathrm{Diff}(M)$.

Key Observation: If $p = 0 \in \mathbb{R}^n$, and $q \in B(1) \subset \mathbb{R}^n$, then $\exists \psi \in \mathrm{Diff}(\mathbb{R}^n)$ such that $\psi(p) = q$ and $\mathrm{supp}(\psi) \subset B(2)$.

[**Remark:** For diffeomorphisms, as they are injective we define their support slightly differently. We define it to be $\mathrm{supp}(\psi) := \overline{\{x \in \mathbb{R}^n : \psi(x) \neq x\}}$, i.e. the closure of the points that ψ moves.]

To prove/see the key observation, note that by symmetry, we can take $q \in \mathbb{R}_{>0} \times \{0\}$ (i.e. positive x-axis coming out of the origin in $\mathbb{R}^n$). Now take a bump function $\chi : \overline{B^n(2)} \to \mathbb{R}$, such that $\chi|_{B^n(1)} \equiv 1$, and $\chi \equiv 0$ outside $\overline{B^n(2)}$.

Then, $\chi \cdot \frac{\partial}{\partial x_1} \in \Gamma(\mathbb{R}^n)$ is a vector field which vanishes outside $B^n(2)$, and as this is a complete vector field [**Exercise** to check - comes from universal existence of solutions to the PDE], take the flow φ_t of this vector field along the x-axis.

Then there will be a time t such that $\varphi_t(0) = q$, since the flow will push $p = 0$ past $B^n(1)$ (along the axis, which q lies on), and we assumed $q \in B(1)$.

So having proven the key ingredient, by continuity of ψ, and rotations, scalings, etc, one can show that both U and U' are open [**Exercise** to check]. Alternatively, just working with the U we see that the sets U for different p give a partition of M into disjoint open non-empty sets, and so by connectedness all most be empty but one, and so done. □

Now we define a more powerful type of bump function which will be very useful.

Definition 1.15. *Let M be a manifold and $\{V_\alpha\}_{\alpha\in A}$ an open cover of M. Then a* ***partition of unity subordinate to the cover*** *is a collection $\{\varphi_\alpha\}_{\alpha\in A}$ of smooth functions, where $\varphi_\alpha : M \to \mathbb{R}$, such that:*

(i) $\varphi_\alpha \geq 0 \ \ \forall \alpha \in A$.

(ii) *$supp(\varphi_i) \subset V_{\alpha_i}$ for some $\alpha_i \in A$.*

(iii) *Supports are locally finite, i.e. $\forall x \in M$, $\exists U \ni x$ open such that only finitely many φ_α are non-zero on U, i.e. $\{i : supp(\varphi_i) \cap U \neq \emptyset\}$ is finite.*

(iv) *$\sum_{\alpha\in A} \varphi_\alpha \equiv 1$ in $C^\infty(M)$ (this prevents $\varphi_\alpha \equiv 0$ for all α).*

Note that by (iii), we know that at each $x \in M$, the sum in (iv) is finite, so is well-defined.

Fact: If M is a manifold, then **every** open cover of M admits a subordinate partition of unity (this is due to M being second countable).

Proof. None given [**Exercise** in point-set topology]. □

So partitions of unity always exist. This enables us to prove that we can always embed smooth manifolds in some overall Euclidean space.

Corollary 1.1 (Whitney Embedding)**.** *Every smooth n-manifold embeds as a submanifold of $\mathbb{R}^{2n}$.*

Proof Sketch. It is easy to prove this for $\mathbb{R}^{2n+1}$, and so we give the argument for this. Some more thought is needed to improve this to $\mathbb{R}^{2n}$.

Suppose for simplicity that M is compact. Then take a finite covering $\{U_i\}_{i\in I}$ of M by charts (i.e. a finite atlas), and a corresponding partition of unity subordinate to this finite covering.

So over U_i, we have a chart map $\varphi_i : U_i \hookrightarrow \mathbb{R}^n$, and a map from the partition of unity, $f_i : U_i \to [0,1]$.

So define $\Phi : M \to \mathbb{R}^N$, where $N = (n+1)|I|$ by

$$\Phi(p) = (f_1 \circ \varphi_1(p), f_1 \circ \varphi_2(p), \ldots, f_1 \circ \varphi_n(p), f_2 \circ \varphi_1(p), \ldots, f_{|I|} \circ \varphi_n(p), f_1(p), \ldots, f_{|I|}(p))$$

which lies in $\mathbb{R}^{n|I|+|I|} = \mathbb{R}^{(n+1)|I|}$.

Then we can check [**Exercise**] that this Φ is injective, and $D\Phi$ is injective, and so it at least yields an immersion. Then we can think about why it is a homeomorphism onto its image, and then how we can improve the dimension by 1 (so get $\mathbb{R}^{2n+1}$ instead of $\mathbb{R}^{2n}$).

□

1.2. **Cotangent Bundles, Lie Algebras and Lie Groups.**

Definition 1.16. *A* ***Lie algebra*** *is a vector space $\mathfrak{g}$ and a bilinear form, called the* ***Lie bracket,*** $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, such that*

(i) $[x, y] = -[y, x]$ *(skew-commutativity)*

(ii) $[x,[y,z]] + [y,[z,x]] + [z,[x, y]] = 0.$ *(Jacobi identity)*

Recall that if M is a manifold, then sections $X \in \Gamma(TM)$ are called vector fields. Each such X also defines a linear operation on function, $C^\infty(M) \to C^\infty(M)$, via $f \mapsto X \cdot f$. If this is in local coordinates $\{x_1,\ldots,x_n\}$ on M, and we write $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$, then we know that

$$X \cdot f = \sum_{i=1}^n X_i \cdot \frac{\partial f}{\partial x_i}$$

and there is an alternative expression in terms of the flow generated by X.

Remark: Given a smooth vector bundle $E \to M$, and a natural operation on vector spaces (e.g. direct sums or dual spaces), we can apply this operation fibrewise to E (as each fibre is a vector space), to build a new bundle, $\tilde{E}$.

So in particular, $\exists$ a bundle $E^* \to M$ formed by applying the dual to each fibre, i.e. $(E^*)_p = (E_p)^*$, called the **dual bundle** of E. In particular, applying this to the tangent bundle $E = TM$, we see that every manifold has a **cotangent bundle**, denoted T^*M (every manifold has one since they all have tangent bundles). So, $T_p^*M = (T_pM)^*$.

So if we have local coordinates $\{x_1,\ldots,x_n\}$ in an open set $U \subset M$, so that $\left.\frac{\partial}{\partial x_i}\right|_q$ span T_qM for $q \in U$, then we denote by $\mathrm{d}x_i|_q$ the dual basis to this basis of T_qM. This gives a basis of $T_q^*M = (T^*M)_q$, the fibre at q of the cotangent bundle.

So as these are the dual basis, we have

$$\mathrm{d}x_i|_q \cdot \left.\frac{\partial}{\partial x_j}\right|_q = \delta_{ij}$$

(where '$\cdot$' means "evaluation at").

Note that a smooth map $f : M \to \mathbb{R}$ defines a global section of T^*M, denoted $\mathrm{d}f$, via:

$$\mathrm{d}f(X) := X \cdot f \quad \text{for} \quad X \in \Gamma(TM),$$

i.e. the section is $s : M \to T^*M$, where $s(p)$ is a map $T_p^*M \to \mathbb{R}$ given by: $(s(p))(X) = (X(p))(f) = (X \cdot f)|_p$.

In local coordinates, by acting on each basis vector of T_qM, we see that

$$\mathrm{d}f_q = \sum_i \left.\frac{\mathrm{d}f}{\mathrm{d}x_i}\right|_q \cdot \mathrm{d}x_i|_q \quad \in \quad T_q^*M.$$

Note: We write: $\Gamma(T^*M) =: \Omega^1(M)$: the **differential 1-forms**. So sections of the cotangent bundle are 1-forms.

So hence if $f : M \to \mathbb{R}$ is smooth, then $\mathrm{d}f \in \Omega^1(M)$. If instead $f : M \to N$, then $\mathrm{d}f : TM \to TN$. Indeed, $\mathrm{d}f_p$ acts on derivations at $p \in M$ and needs to give a derivation at $f(p) \in N$. This is done by:

$$\boxed{\left(\mathrm{d}f_p(v)\right)(g) := v(g \circ f)}$$

for $g \in C^\infty(N)$ and $v \in T_pM$. We also call $\mathrm{d}f$ the **pushforward**, and denote it by f_*.

Note that clearly a vector field is completely determined by the map $C^\infty(M) \to C^\infty(M)$, $f \longmapsto X \cdot f$ (as X is a finite linear combination of the $\frac{\partial}{\partial x_i}$).

Definition 1.17. *If $X, Y \in \Gamma(TM)$ are vector fields, then their* ***commutator*** *$[X,Y]$ is defined to be the linear map $C^\infty(M) \to C^\infty(M)$ defined by:*

$$[X,Y] \cdot f := X \cdot \underbrace{(Y \cdot f)}_{\in C^\infty(M), \text{ so } X \text{ can act on it}} - Y \cdot (X \cdot f)$$

i.e. $[X,Y] := XY - YX$.

Lemma 1.6. *$[X,Y] \in \Gamma(TM)$, i.e. it is a vector field. In particular, $\forall p \in M$, $[X,Y]|_p$ is a derivation of T_pM.*

Proof. If locally we have $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^n Y_j \frac{\partial}{\partial x_j}$, then one can compute:

$$[X,Y] = \sum_i \underbrace{\left(\sum_j \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j}\right)\right)}_{=:A_i} \frac{\partial}{\partial x_i} = \sum_i A_i \frac{\partial}{\partial x_i}$$

i.e. the terms involving $\frac{\partial^2 f}{\partial x_i \partial x_j}$ in $[X,Y] \cdot f$ cancel. This shows (as this is just some sum of the $\frac{\partial}{\partial x_i}$), that this is a derivation.

□

Properties of commutator: From the local expression of $[X,Y]$ as found in the above proof of Lemma 1.6, one can check the following properties of the commutator:

(i) $[X,Y] = -[Y,X]$

(ii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

(iii) $[fX, gY] = f \cdot g[X,Y] + f(X \cdot g)Y - g(Y \cdot f)X$, where the product fX is defined pointwise, since $\Gamma(TM)$ has a $C^\infty(M)$-module structure (i.e. this will give another vector field).

(iv) If $\varphi : M \to N$ is smooth, then it induces a map $\mathrm{d}\varphi : TM \to TN$, and then:

$$\mathrm{d}\varphi([X_1, X_2]) = [\mathrm{d}\varphi(X_1), \mathrm{d}\varphi(X_2)]$$

i.e. smooth maps give **Lie algebra homomorphisms on tangent spaces**.

Note: Properties (i) and (ii) above tells us that $(\Gamma(TM), [\cdot,\cdot])$ is an infinite dimensional Lie algebra.

Proof of Property (iv). If $\varphi : M \to N$ is smooth, then $\mathrm{d}\varphi|_p : T_pM \to T_{\varphi(p)}N$. So $\mathrm{d}\varphi_p(X_p) \in T_{\varphi(p)}N$. Hence $\mathrm{d}\varphi(X)$ can only be evaluated at points in the image $\varphi(M)$ (as it may not hit everything), i.e. we have:

$$\mathrm{d}\varphi(X)|_{\varphi(p)} := \mathrm{d}\varphi|_p(X_p).$$

If φ were a diffeomorphism, this expression take the more recognisable form of

$$\mathrm{d}\varphi(X)|_p := \mathrm{d}\varphi|_{\varphi^{-1}(p)}\left(X_{\varphi^{-1}(p)}\right).$$

So hence we have:

$$\mathrm{d}\varphi([X_1,X_2])|_{\varphi(p)}(f) = [X_1,X_2]|_p(f\circ\varphi).$$

For the RHS of (iv) we have:

$$\begin{aligned}[\mathrm{d}\varphi(X_1).\mathrm{d}\varphi(X_2)]\,|_{\varphi(p)}(f) &= \mathrm{d}\varphi(X_1)|_{\varphi(p)}\cdot(\mathrm{d}\varphi(X_2)(f)) - \ (1 \longleftrightarrow 2)\\ &= X_1|_p \cdot \left(\mathrm{d}\varphi(X_2)|_{\varphi(\cdot)}(f)\right) - \ (1 - \longleftrightarrow 2)\\ &= X_1|_p \cdot (X_2\cdot(f\circ\varphi)) - \ (1 \longleftrightarrow 2)\\ &= [X_1,X_2]|_p\cdot(f\circ\varphi)\\ &= \mathrm{d}\varphi([X_1,X_2])|_{\varphi(p)}(f)\end{aligned}$$

where in the second line, $\mathrm{d}\varphi(X_2)|_{\varphi(p)}(f)$ is the smooth function on M defined by: $p \mapsto \mathrm{d}\varphi(X_2)|_{\varphi(p)}(f) \equiv X_2|_p\cdot(f\circ\varphi)$.

□

Note: Another common notation for $\mathrm{d}\varphi$ as in property (iv) above is: $\varphi_* : TM \to TN$, and so property (iv) becomes:

$$\varphi_*([X_1,X_2]) = [\varphi_*(X_1), \varphi_*(X_2)].$$

The dual map of φ_*, $\delta\varphi : T^*N \to T^*M$ (as $\mathrm{d}\varphi$ is a linear functional, and so we can consider its dual), is usually denoted φ^*. The upper $*$'s mean **contravariant**. Lower $*$'s mean **covariant** (see Category Theory. This is because d is covariant, i.e. $\mathrm{d}(\varphi\circ\psi) = \mathrm{d}(\varphi)\circ\mathrm{d}(\psi)$).

Definition 1.18. *A **Lie group** G is a manifold which is also a group, such that the group operations of multiplication, $m : G\times G \to G$, and inversion $G \to G$, $g \longmapsto g^{-1}$, are smooth maps (with respect to the appropriate topologies).*

Example 1.9. *Consider $GL_n(\mathbb{R}) \subset Mat_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ (an open subset). This is a smooth manifold, and a group, and hence is a Lie group, under the usual matrix multiplication. Similarly, $O(n) \subset GL_n(\mathbb{R})$, the orthogonal group, is a Lie group, and indeed, it is a closed Lie subgroup of $GL_n(\mathbb{R})$.*

Lie groups are interesting because they have **globally** trivial tangent bundles. To see this, let G be a Lie group, and $g \in G$. Then let $L_g : G \to G$ be the left multiplication/translation by g, i.e. $L_g(h) = gh$. Note $e \mapsto g$ under L_g.

Then, L_g is a self-diffeomorphism of G (as the inverse is $L_g^{-1} = L_{g^{-1}}$). Thus taking tangent spaces at the origin, we have

$$\mathrm{d}L_g\big|_e : T_eG \xrightarrow{\cong} T_{L_g(e)}G = T_gG$$

is an isomorphism. So via these maps, we see that **all** tangent spaces T_gG can be canonically identified with the fixed vector space T_eG.

This shows that the map $TG \to G \times \mathbb{R}^r \cong G \times T_eG$, where $r = \dim(T_eG)$, defined via

$$\left(DL_g\big|_e\right)^{-1}(\xi) \longleftrightarrow (g, \xi),$$

defines a **global trivialisation** of TG.

Definition 1.19. *If $X \in \Gamma(TG)$ with G a Lie group, then we say that X is **left-invariant** if $\forall g \in G$, $DL_g(X_h) = X_{gh}$.*

Here, $X_h = X(h) \in T_hG$. So this can be thought of as, for example, a sphere with the vector field being independent in one of the angle variables, and we rotate in that angle, we get the same thing back, i.e. X is mapped to itself under the action of g (i.e. the vector field where h is mapped to, gh, is the just that from acting on the one at h). So hence left-invariant vector fields are completely determined by one point, as its images under $\mathrm{D}L_g$ give the rest (as the action is transitive).

Hence the way to think of left-invariant vector fields is that, for any vector $v \in T_eG$, we can generate a left-invariant vector field via acting on v by each $\mathrm{D}L_g$ for each $g \in G$ (i.e. by each group element in some sense) to determine the value of the vector field at g. Hence for each $v \in T_eG$ we get a left-invariant vector field in this way, and each such vector field is completely determined by its value at e, i.e. $X(e) \in T_eG$ (as the action is transitive). So hence we have a 1-1 correspondence between these objects, and so we get a 4th equivalent description of the tangent bundle (of a Lie group).

Let $\mathrm{Vect}_L(G) \subset \Gamma(TG)$ be the **subspace of left-invariant vector fields**. Then property (iv) of the commutator shows that:

$$\mathrm{d}L_g([X,Y]) = [\mathrm{d}L_g(X), \mathrm{d}L_g(Y)]$$

and so we see that $\mathrm{Vect}_L(G)$ is a Lie subalgebra of $\Gamma(TG)$. Moreover, this is a finite dimensional Lie subalgebra, of dimension $\dim(T_eG)$ (since each left-invariant vector field it is determined by one point, which lies in T_eG. So hence the dimension is just the dimension of T_eG).

Notation: Write $\mathfrak{g} := T_eG \cong \mathrm{Vect}_L(G)$ for this Lie algebra.

So hence for every Lie group, we get an associated finite dimensional Lie algebra, which is just the tangent space T_eG, or equivalently $\mathrm{Vect}_L(G)$.

Lemma 1.7. *Let G be a Lie group, with Lie algebra $\mathfrak{g} = T_e(G)$. Let $\xi \in \mathfrak{g}$. Then ξ defines a left-invariant vector field $X_\xi \in \Gamma(TG)$, which is complete (i.e. a globally defined flow).*

(So this in particular defines a 1-parameter subgroup $\mathbb{R} \to G$, via the maximal integral curve X_ξ through $e \in G$.)

Proof. By the above, we know that any such ξ determines a left-invariant vector field. So it suffices to just show that this vector field is complete.

If a maximal integral curve γ was defined on some finite interval $(-q_1, q_2)$, then by translating by $g = \gamma(t)$ for t close to q_2 and using left-invariance, we obtain an extension of the flow line (i.e. translate back to e, where we know how the flow works locally, and extend by that. By left-invariance this flow should be the same as that at $\gamma(t)$),

i.e. if φ_t is our flow, that $\varphi_t(hg) = h\varphi_t(g) = L_h(\varphi_t(g))$, i.e. $\varphi_t \circ L_h = L_h \circ \varphi_t$.

□

This leads us straight to the definition of the **exponential map:**

Corollary 1.2. *There is a uniquely defined map, called the **exponential map**,* $\exp : \mathfrak{g} \to G$*, with* $\mathfrak{g}, G$ *as above, such that*

(i) $t\xi \mapsto \gamma_\xi(t)$

(ii) $\mathrm{d}(\exp)|_0 = id$,

where γ *is the flow from above, and* $t \mapsto (\exp(\xi))(t)$ *is the unique 1-parameter subgroup of* G *with tangent vector* ξ *at* $0 \in \mathbb{R}$ *(as above),*

i.e. we can take this to be a subgroup of G*, which has a global extension (and then follow the extension to get the map).*

Proof. Defining the map as in the statement of the corollary, we have (writing $\sigma(t) := t\xi$):

$$\mathrm{d}\exp|_0(\xi) = \mathrm{d}\exp|_0(\sigma'(0)) = \mathrm{d}(\mathrm{d} \circ \sigma)|_0 = \mathrm{d}(\exp(t\xi))|_{t=0} = \mathrm{d}(\gamma_\xi(t))_{t=0} = \gamma'_\xi(0) = \xi.$$

□

Note: The exponential map above is understood in the following way. For each $\xi \in \mathfrak{g}$, by Lemma 1.7, we get a corresponding complete vector field on G. We can then integrate this vector field to get a flow, defined for all time by completeness of the vector field. Thus we get a map on $G \to G$ determined by the flow up to time t, which is our 1-parameter subgroup due to the "$\gamma_{s+t} = \gamma_s \circ \gamma_t$" relation. However the exponential map is in particular interested in the image of e at time 1 of this flow, i.e. $\exp(\xi) = \gamma_\xi(1)|_e$. The above shows that $\gamma_{t\xi}(1) = \gamma_\xi(t)$.

Remark: exp is smooth as you vary any of the parameters (this is just by the smoothness of solutions to such ODEs). The inverse function theorem also says (since $\mathrm{d}(\exp)|_0 = \mathrm{id}$ is invertible) that exp defines a local diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ to a neighbourhood of $e \in G$.

Example 1.10. *Consider $G = GL_n(\mathbb{R})$. Then if $A \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) \cong \mathbb{R}^{n^2} \cong Mat_n(\mathbb{R})$, then we can find:*

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots =: e^A$$

i.e. this gives us the usual exponential of a matrix (hence the name "exponential map"!)

[This is true since if we consider $t \mapsto e^{tA} := I + tA + \cdots$, then this gives a 1-parameter subgroup of $GL_n(\mathbb{R})$, namely $\{e^{tA}\}_{t\in\mathbb{R}}$, with the correct tangent vector at $0 \in \mathbb{R}$ (which is A), and so by uniqueness of the exponential map, by checking the other property we get that this must be exp*. Note how at $t = 1$ of this flow, e (which here is the identity matrix, I, which is the value at $t = 0$ of the subgroup) is mapped to* $\exp(A)$*.]*

Remark: If $\varphi : G \to H$ is a smooth map of Lie groups which is also a homomorphism, then we obtain a commutative diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\varphi"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\varphi"'] & H. \end{tikzcd}$$

This is because $t \mapsto \varphi(\exp(t\xi))$ **is** a 1-parameter subgroup with the correct tangent vector, and so by uniqueness of the exponential map, this must come from exp in H. Hence the two maps must be the same.

Indeed, we want to show that $\tilde{\gamma}(t) := \varphi(\exp(t\xi)) \equiv \varphi(\gamma_\xi(t))$ is an integral curve through $e \in H$ in the direction of $d\varphi(\xi) \in \mathfrak{h}$. Then we would have:

$$\exp(d\varphi(\xi)) := \tilde{\gamma}(1) = \varphi(\gamma_\xi(1)) = \varphi(\exp(\xi))$$

i.e. $\exp \circ d\varphi = \varphi \circ \exp$. So to see this, note that since γ_ξ is an integral curve,

$$\tilde{\gamma}'(t) = d\varphi_{\gamma_\xi(t)} \cdot \gamma_\xi'(t) = d\varphi_{\gamma_\xi(t)} \cdot (X_\xi)_{\gamma_\xi(t)} = (d\varphi(X_\xi))|_{\varphi(\gamma_\xi(t))} = d\varphi(X_\xi)|_{\tilde{\gamma}(t)}.$$

Thus this shows that $\tilde{\gamma}$ is an integral curve with the correct tangent vector and so we are done.

$\square$

Remarks: A lot more is true on all of these things. Although we cannot go into too much detail here, we shall list some of the interesting properties:

(i) If G is a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, then $\exists!$ connected Lie subgroup $H \subset G$ such that $T_eH = \mathfrak{h}$.

(ii) If G, H are Lie groups and G is simply connected, then any Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ "exponentiates" to a homomorphism $G \to H$.

1.3. **Integrability.**

Let $f : M \to N$ be smooth. Then f induces a map $dd : TM \to TN$ via: $df_p : T_pM \to T_{f(p)}N$. We write $f_* = df$. Taking the dual map of f_*, we get $f^* : T^*N \to T^*M$.

Then given $\alpha \in \Omega^1(N) = \Gamma(T^*N)$ a differential 1-form, we can form $f^*\alpha \in \Omega^1(M)$, where

$$(f^*\alpha)(p) = \alpha(f(p)) \circ \mathrm{d}f_p$$

i.e. $(f^*\alpha)_p = \alpha_{f(p)}(\mathrm{d}f_p)$. As we are dealing with elements of the cotangent bundle which are linear maps, we tend to denote $\alpha(p)$ by α_p, so that the function notation $\alpha_p(v)$ is clearer. So hence $p \in M$ is mapped under $f^*\alpha$ to the composition:

$$T_pM \xrightarrow{\mathrm{d}f} T_{f(p)}N \xrightarrow{\alpha_{f(p)}} \mathbb{R},$$

which is an element of T_p^*M. So we naturally get an induced section/1-form $f^*\alpha \in \Omega^1(N)$.

However given $X \in \Gamma(TM)$ a vector field, f_*X is not in general in $\Gamma(TN)$. Really, f induces a new bundle over M via pulling back the one over N. Indeed, this new bundle, denoted $f^*(TN)$, has for $p \in M$,

$$(f^*(TN))_p := (TN)_{f(p)}$$

fibrewise, and clearly $f_*X \in \Gamma(f^*(TN))$ (as $X_p \in T_pM$, and $f_* = \mathrm{d}f$ maps T_pM to $T_{f(p)}N = (f^*(TN))_p$).

Also, if $Y \in \Gamma(TN)$, then Y also defines a section of $f^*(TN)$ over M. Indeed, define for $p \in M$,

$$f^*(Y)_p = Y_{f(p)} \in T_{f(p)}N = (f^*(TN))_p.$$

So hence for vector fields $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$, we get induced vector fields on $f^*(TN)$, via either pulling back or pushing forward. Hence we define:

Definition 1.20. *We see that vector fields X over M and Y over N are **f-related** if*

$$f_*(X) = f^*(Y) \quad \textit{in} \quad \Gamma(f^*(TN)),$$

i.e. if these induced vector fields agree (i.e. pushing X forward via f is exactly Y).

We have seen that $f_*[X_1, X_2] = [f_*X_1, f_*X_2]$, which really means as sections of $f^*(TN)$, if $X_1$is f-related to Y_1 and X_2 is f-related to Y_2, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$, as then

$$f_*[X_1, X_2] = [f_*X_1, f_*X_2] = [f^*Y_1, f^*Y_2] = f^*[Y_1, Y_2]$$

where the last equality is easily seen by definition of f^*Y.

Remark: In general, $X \in \Gamma(TM)$ need not be f-related to any $Y \in \Gamma(TN)$, and even if it is, the Y need not be unique.

But: If $f : M \to M$ (i.e. $N = M$) is a diffeomorphism, then f_*X is [**Exercise** to check] a perfectly well-defined vector field on M.

So the moral is, we can push forward vector fields by diffeomorphisms, but we need to be more careful with other maps.

There is a geometrical interpretation for when two vector field commute, as shown in the following lemma:

Lemma 1.8. *Let M be a manifold and $X, Y \in \Gamma(TM)$. Then:*

$$[X,Y]=0 \iff \text{The local flows defined by } X \text{ and } Y \text{ commute.}$$

Proof. The key observation if that

$$[X,Y]|_p = \lim_{t\to 0} \frac{Y|_p - (\varphi_t)_* Y|_p}{t},$$

where φ_t is the flow induced by X.

Given this, then we have:

$$(\dagger) \qquad \begin{aligned} [X,Y]=0 &\iff Y \text{ is invariant under the flow of } X \text{ (and vice versa by symmetry)} \\ &\iff \text{The flows commute} \end{aligned}$$

(see the remark after the proof for more explanation). So we just need to prove this key observation. So let $f \in C^\infty(M)$, and write

$$f_t := f \circ \varphi_t = (\varphi_t)_*(f).$$

Then we know from a previous result,

$$X \cdot f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_t)_*(f),$$

and so we can write (using a Taylor series expansion)

$$f_t = f + t(X\cdot f) + t^2 h_t$$

for some smooth function h_t. Also, we know

$$(\varphi_t)_*\,(Y)|_p\,(f) = Y|_{\varphi_t^{-1}(p)}(f\circ\varphi_t) = Y|_{\varphi_t^{-1}(p)(f_t)}.$$

So combining all of this, we have:

$$\begin{aligned}
\lim_{t\to 0}\frac{Y|_p - (\varphi_t)_*(Y)|_p}{t}(f) &= \lim_{t\to 0}\frac{Y|_p(f) - (\varphi_t)_*(Y)|_p(f)}{t} = \lim_{t\to 0}\frac{Y|_p(f) - Y|_{\varphi_t^{-1}(p)}(f_t)}{t} \\
&= \lim_{t\to 0}\frac{Y|_p(f) - Y|_{\varphi_t^{-1}(p)}(f + t(X\cdot f) + t^2 h_t)}{t} \qquad \text{(by Taylor expanding } f_t\text{)} \\
&= \lim_{t\to 0}\frac{Y|_p(f) - Y_{\varphi_t^{-1}(p)}(f)}{t} - Y_p\cdot(X\cdot f) \quad \text{(dealing with bits linear in } t \text{ or higher)} \\
&= X|_p\cdot(Y\cdot f) - Y|_p\cdot(X\cdot f) \quad \text{(by definition of } X\cdot g\text{, with } g = Y\cdot f\text{)} \\
&= [X,Y]|_p(f).
\end{aligned}$$

Then since this was true for general f, we are done with proving the key observation and thus are done with the proof.

□

Remark: Seeing the equivalences in (†) requires more explanation. In general, let $\alpha : M \to N$ be a diffeomorphism and $X \in \Gamma(TM)$ a vector field with flow φ_t. Then $\alpha_* X$ has flow $\alpha\circ\varphi_t\circ\alpha^{-1}$ (i.e.

map back to M, flow, then map back), since:

$$\alpha_* X|_q(f) = X_{\alpha^{-1}(q)}(f\circ\alpha) = \lim_{t\to 0}\frac{(f\circ\alpha)(\varphi_t(\alpha^{-1}(q))) - (f\circ\alpha)(\alpha^{-1}(q))}{t}$$
$$= \lim_{t\to 0}\frac{f\circ\left(\alpha\circ\varphi_t\circ\alpha^{-1}\right)(q) - f(q)}{t}.$$

With this we see that $\alpha_* X = X \iff \alpha\circ\varphi_t\circ\alpha^{-1} = \alpha \iff \alpha\circ\varphi_t = \varphi_t\circ\alpha$.

With this we can show tat $[X,Y]=0$ if and only if the flows of X,Y commute. Indeed, the backwards direction ($\Leftarrow$) is simple, as then as φ_t is a diffeomorphism $M\to M$ by the above we get $(\varphi_t)_* Y = Y$, and so (from the expression for $[X,Y]|_p$ proven in Lemma 1.8) we see $[X,Y]=0$.

For the other direction ($\Rightarrow$), define the curve $c(t)$ by: $c(t) := ((\varphi_t)_* Y)_p$. Then we can show that $c'(t) = (\varphi_t)_*(0) = 0$, and so c is constant. Hence $c(t) = c(0)$ for all t, i.e. $(\varphi_t)_* Y = Y$, and so $\varphi_t\circ\psi_s = \psi_s\circ\varphi_t$ for all t,s, i.e. the flows commute.

□

Remark: Suppose $\mathfrak{g}$ is a finite dimensional abelian Lie algebra, i.e. $\mathfrak{g}\cong\mathbb{R}^n$, and $[\cdot,\cdot]\equiv 0$. Let G be a Lie group with $T_eG\cong\mathfrak{g}$ (which we can find from a fact from before).

Then if we take a basis $\xi_1,\dots,\xi_n$ for T_eG, we get associated flows X_ξ on G which commute (as the Lie algebra is abelain). In particular, $\exp:\mathfrak{g}\to G$ becomes a homomorphism of abelian Lie groups (i.e. exp is surjective onto a neighbourhood of $e\in G$, and hence if all points commute, the group structure $\implies G$ is ableian, i.e. locally abelian $\implies$ globally abelian).

Fact from topology: If G is a connected Lie group, then a subgroup which contains an open neighbourhood of $e\in G$ must be all of G, and therefore G is a quotient of a vector space.

Then, if G is also compact, this implies that $G\cong T^n$ must be a torus (as the vector space will be $\cong\mathbb{R}^n$ for some n, and when quotienting we will get $\mathbb{R}^{n-k}\times T^k$ for some k. Thus to be compact we would need $n-k$).

In the above proof, we used the fact that: $X\cdot f = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\varphi_t)_* f$. However, we can differentiate more or less anything along the flow lines of a vector field. This motivates the following definitions.

Definition 1.21. *If M is a manifold, then a* ***(k,l)-tensor on M*** *is a section of the bundle: $(T^*M)^{\otimes k}\otimes(TM)^{\otimes l}$.*

Here, $V^{\otimes k} := \bigotimes_{i=1}^k V$. So hence we see that a $(1,0)$-tensor is simply a (differential) 1-form, and a $(0,1)$-tensor is simply a vector field.

Definition 1.22. *The* ***Lie derivative*** *of a tensor τ along a vector field X is:*

$$\mathfrak{L}_X\tau := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\varphi_t)_*\tau$$

where φ_t is the natural map induced by $\varphi_* : TM \to TM$ and $\varphi^* : T^*M \to T^*M$ on (k,l)-tensors (i.e. act on each component by either φ^* or φ_*).

Example 1.11. *We give some examples/properties of the Lie derivative to enable easier calculation:*

(i) If $g \in C^\infty(M)$ is a (0,0)-tensor, then $(L)_X g = X \cdot g$, as proven in Proposition 1.1.

(ii) If Y is a vector field, then $\mathfrak{L}_X(Y) = [X,Y]$ (this is what we just proved in Lemma 1.8, as $\varphi_0 = id$).

(iii) We always have, from skew-commutativity of the Lie bracket,

$$\mathfrak{L}_X(Y) = -\mathfrak{L}_Y(X),$$

and

$$\mathfrak{L}_X([Y,Z]) = [\mathfrak{L}_X(Y),Z] + [Y,\mathfrak{L}_X(Z)]$$

which is a Lie derivative version of the Jacobi identity (this gives the Jacobi identity in case (2)).

Example 1.12. *There is a rank 2 subbundle of $T\mathbb{R}^3$, spanned by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ at a point $(x,y,z) \in \mathbb{R}^3$. What this looks like can be found by searching "Contact structure" on Google Images.*

Exercise: Show that there is no surface $\Sigma \subset \mathbb{R}^3$ with $0 \in \Sigma$ such that:

$$T_p\Sigma = \left\langle \frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\rangle\Big|_p$$

for every $p \in \Sigma$. Contrast this with the result that any 1-dimensional subbundle, i.e. a vector field, **is** tangent to a family of curves. □

So what this example and exercise shows is that with higher dimensional tensors, we cannot necessarily find surfaces, etc which are tangent to the tensor at every point. However in the vector field case, we could (via integrating the flow to find a path, i.e. integral curves). We will now work towards a result which tells us when we can find such a surface/curve.

Definition 1.23. *Let M be a manifold and $E \subset TM$ be a subbundle of rank k. Then E is* ***involutive*** *if it is closed under the Lie bracket, i.e.*

$$\forall X, Y \in \Gamma(E) \subset \Gamma(TM), \text{ we have: } [X,Y] \in \Gamma(E).$$

Definition 1.24. *A subbundle of TM is called a* ***distribution****.*

Definition 1.25. *Let E be a distribution and $N \subset M$ be a submanifold. We say that E is an* ***integral submanifold of E*** *if for all $p \in N$ we have $T_pN = E_p$.*

We say that E is ***integrable*** *if at each point of M $\exists$ an integrable submanifold of E.*

Theorem 1.3 (The Frobenius Integrability Theorem (F.I.T)). *Let M^n be a manifold, and E^k an involutive distribution of rank k. Then, $\forall p \in M$, $\exists$ local coordinates $\{x_1, \ldots, x_k\}$ near p such that $\left\{\frac{\partial}{\partial x_1} \ldots, \frac{\partial}{\partial x_k}\right\}$ is a local basis of sections of E.*

i.e. it is locally ***flat****, as we have this nice basis.*

Note: When the latter condition occurs we say that E is **integrable**. So the F.I.T tells us that involutive distributions are integrable. [Integrable just means that we can 'integrate' local coordinates on a manifold to get a local basis of sections of E. The idea is that, as usual, using flows we get hit everything.]

Remark: If $p \in Y^k \subset M^n$ is a k-dimensional submanifold, then $\Gamma(TY) \subset \Gamma(TM)$ is a sub Lie algebra (i.e. $[Y_1, Y_2] \in \Gamma(TY)$ for all $Y_1, Y_2 \in \Gamma(TM)$). So locally, $M \cong Y \times \mathbb{R}^{n-k}$ (i.e. M is **foliated** by k-dimensional submanifolds). Then the associated subbundle given by $TY \subset TM$ is involutive.

So hence the above therefore, suitably interpreted, is an "$\Longleftrightarrow$".

Proof of F.I.T. Start with the following special case.

Suppose locally near $p \in M$, we have vector fields $\hat{E}_1, \ldots, \hat{E}_k \in \Gamma(E) \subset \Gamma(TM)$ forming a local basis for E, such that $[\hat{E}_i, \hat{E}_j] = 0$.

Then near p, we can find an open neighbourhood U of $0 \in \mathbb{R}^k$ ($\cong (-\varepsilon, \varepsilon)^k$ for some $\varepsilon > 0$) and a smooth map $F : U^k \to M^n$, with $F(0) = p$, by letting

$$F(y_1, \ldots, y_k) = \left(\varphi_{\hat{E}_1, y_1} \circ \cdots \circ \varphi_{\hat{E}_k, y_k}\right)(p).$$

Here, $\hat{E}_i$ has associated flow $\varphi_{\hat{E}_i, t}$, and we are flowing for time given by the coordinates $\{y_i\}_i$.

Note that if ε is sufficiently small, this makes sense, i.e. the images $\varphi_{\hat{E}_j, y_j}$ stay where the next flow is defined, **and** the flows commute (as the E_i commutes, using Lemma 1.8). So for each j, by commuting the flows, we have

$$F(y_1, \ldots, y_k) = (\varphi_{\hat{E}_j, y_j}) \circ (\varphi_{\hat{E}_1, y_1} \circ \cdots \circ \underbrace{\hat{\varphi}_{\hat{E}_j, y_j}}_{\text{omit}} \circ \cdots \circ \varphi_{\hat{E}_k, y_k})(p).$$

So hence differentiating with respect to y_j gives:

$$\left.\frac{\partial F}{\partial y_j}\right|_y = \hat{E}_j\big|_F,$$

since these are integral curves. This tells us that $DF|_y$ sends $\frac{\partial}{\partial y_j} \longmapsto \hat{E}_j$ in this neighbourhood. So since $\hat{E}_1, \ldots, \hat{E}_k$ are linearly independent, we see that $DF|_p$ is injective, and so shrinking ε if necessary (so to use the inverse function theorem), F is locally a diffeomorphism onto its image, and so we are done.

So hence this all works when the vector fields $\hat{E}_j$ commutes.

In the general case, at $p \in M$, we pick a coordinates chart U such that:

$$E|_p = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \rangle$$

holds **at p**. Then since E is smooth, for y in a neighbourhood of p, we can find bases of sections of E_y of the form:

$$\left\{ \frac{\partial}{\partial x_i} + \sum_{j=k+1}^{n} a_{ij}(y) \frac{\partial}{\partial x_j} \right\}_i$$

for $a_{ij}(y)$ locally defined smooth functions, vanishing for $y = p$ (we can do this by a (smooth) version of Gram-Schmidt) [Recall that k was the rank of E]. So let

$$\hat{E}_i\Big|_y := \frac{\partial}{\partial x_i} + \sum_{j=k+1}^{n} a_{ij}(y) \frac{\partial}{\partial x_j}.$$

Then the key point is that since $\forall i, j$ we have $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$, we get that

$$\left[\hat{E}_i, \hat{E}_j\right] \in \operatorname{span}\langle \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n} \rangle.$$

But then by hypothesis, E is an involutive distribution, and so:

$$\left[\hat{E}_i, \hat{E}_j\right] \in \Gamma(E) = \operatorname{span}\langle \hat{E}_1, \ldots, \hat{E}_k \rangle$$

locally. But then by considering the coefficients of $\frac{\partial}{\partial x_j}$ for $j \leq k$, we can see that we must have $\left[\hat{E}_i, \hat{E}_j\right] = 0$.

Hence these commute, and so by the previous case we are done.

$\square$

Good Exercise: Prove the F.I.T by induction on $k = \operatorname{rank}(E)$.

Note: If G is a Lie group, then each Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} = T_e G$ defines an involutive distribution (from what we have previously seen, i.e. those vector fields generated by $v \in \mathfrak{h}$). The corresponding integral submanifold of G through e and tangent to $\mathfrak{h}$ is a connected Lie subgroup of G.

Next we prove a result about quotient manifolds, using the F.I.T to make sure we have nice bases.

Corollary 1.3. *Let M be a smooth manifold and G a compact Lie group, which acts freely on M. Then, M/G is a manifold.*

Recall: By 'acting freely' we mean only the identity element fixed all of M under the group action.

Example 1.13. *Let $G_k(\mathbb{R}^n) = \{k$-dimensional subspaces of $\mathbb{R}^n\}$ (=Grassmannians). So, for example if $k = 1$, then $G_1(\mathbb{R}^n) \cong \mathbb{R}P^{n-1}$. Then by choosing bases for the subspaces and their orthogonal complements, relating bases by elements of $O(n)$, we see:*

$$G_k(\mathbb{R}^n) \cong \frac{O(n)}{O(k) \times O(n-k)},$$

and so by the above Corollary 1.3, this is a manifold.

Example 1.14 (Non-Example)**.** *Let $G = \mathbb{R}$ act on $S^1 \times S^1 =: M$, via: $(\theta_1, \theta_2) \mapsto (\theta_1 + c_1 t, \theta_2 + c_2 t)$ on angle coordinates, for some constants c_1, c_2 such that $c_1/c_2 \notin \mathbb{Q}$.*

Then M/G is not Hausdorff, and so cannot be a manifold.

Proof of Corollary 1.3. We start with some topology.

The projection map $M \to M/G$, being a quotient map, is by definition an open map. We claim that is is also a closed map.

So suppose $A \subset M$ is closed. We want to show that $\pi(A)$ is closed. Then define:

$$\pi(A) \equiv G \cdot A := \bigcup_{g \in G} gA$$

the action of G on A, and observe that,

$$x \in M \backslash G \cdot A \iff G \times \{x\} \cap \varphi^{-1}(A) = \emptyset,$$

where $\varphi : G \times M \to M$ is the action of G on M. Note $x \in G \cdot A$ if and only if $\exists g \in G, a \in A$ such that $g \cdot a = x$. Now, $\varphi^{-1}(A) = \{(g, m) : g \cdot m \in A\}$ is closed, and so we can find open $U_g \subset G$, $V_g \subset M$, such that $x \in V_g$ and $(U_g, V_g) \cap \varphi^{-1}(A) = \emptyset$. So let $U_{g_1}, \ldots, U_{g_q}$ be a finite subcover of G (as these cover G and G is compact). Then, we know:

$$x \in \bigcap_{j=1}^{q} V_{g_j} \subset M \backslash G \cdot A,$$

and so hence $M \backslash G \cdot A$ is open, i.e. $G \cdot A =: \pi(A)$ is closed.

Now, if $\pi : M \to M/G$ and $\pi(x) \neq \pi(y)$, then Gx, Gy are disjoint compact subsets of the Hausdorff space M. Then as this is a normal topological space, $\exists$ disjoint open neighbourhoods $U \supset Gx$, $V \supset Gy$ seperating Gx and Gy (note that M/G is the set of equivalence classes, Gx for $x \in M$).

Their projections onto M/G then show that M/G is a Hausdorff space.

Moreover, if $f : X \to Y$ is any closed map, which is surjective and such that $f^{-1}(y)$ is compact $\forall y \in Y$, then: X 2nd countable $\Rightarrow$ Y is 2nd countable.

So if $B = \{U_n\}_{n \in \mathbb{N}}$ is a countable basis for the topology on X, such that B is closed under finite unions (we can always extend a topology to one like this, by adding in all finite unions, etc, which is still countable). Then, $B' = \{Y \backslash f(X \backslash U_n) : n \in \mathbb{N}\}$ forms a countable basis for the topology on Y (as f is a closed map) [**Exercise** to check].

So we want M/G to be a manifold. We need to define neighbourhoods, which is where the action of G and the F.I.T come in.

Note that the action of G on M defines a map $\rho : \mathfrak{g} \to \Gamma(TM)$ (where $\mathfrak{g}$ = the Lie algebra of G). So let $\mathfrak{g} = \langle \xi_1, \ldots, \xi_n \rangle$, and $X_i = \rho(\xi_i)$. These are pointwise linearly independent, since if $\left(\sum_i c_i X_i\right)\big|_p = 0$, then $\xi = \sum_i c_i \xi_i \in \mathfrak{g}$ generates a 1-parameter subgroup of G fixing p, a contradiction to acting freely.

[**Remark:** In general if a vector field X defines a flow φ_t, then:

$$X|_p = 0 \iff \varphi_t(p) = p$$

i.e. p is a fixed point of the flow.]

But we know that G acts **freely**. So if $E = \text{span}\langle X_1, \ldots, X_m \rangle \subset TM$, then this defines a $\dim(G) = \text{rank}(E) = m$-dimensional subbundle, which is involutive, as ρ is a homomorphism of Lie algebras. Indeed:

$$[X_i, X_j] = [\rho(\xi_i), \rho(\xi_j)] = \rho(\underbrace{[\xi_i, \xi_j]}_{\in \mathfrak{g}}) \in E$$

So at $p \in M$, by the F.I.T, we can find local coordinates $y_1, \ldots, y_n$ such that $E = \left\langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m} \right\rangle$. We now define a local "slice" to the action of G by:

$$W = \{y_i = a_i : 1 \le i \le m\}$$

for some constants $a = (a_1, \ldots, a_m)$.

We want to use W as a chart for M/G. So define $F : G \times W \to M$ by $(g, y) \mapsto g \cdot y$, to be the action map restricted to W.

Then by construction, DF is locally a diffeomorphism, and the inverse function theorem gives that $\exists$ open neighbourhoods $U \ni e$ in G, $V \ni y$ in W, such that F is a diffeomorphism on $U \times V$.

> **Claim:** If V is chosen sufficiently small, then $\pi|_V : V \to M/G$ is injective. Thus this defines a chart for M/G near $\pi(y) \in M/G$.
>
> *Proof of Claim.* If not, then $\exists (y_n)_n \subset W$ with $y_n \to y$ and $(g_n)_n \subset G$ with $g_n \to g_\infty$ (as G is compact so can extract convergent subsequence), such that $g_n y_n \in W$ and $g_n y_n \to y$.
>
> Now, $g_n y_n \to g_\infty y \Rightarrow g_\infty = e$ (the identity element of G).
>
> But near $e \in G$, we know that F is a diffeomorphism, and so:
>
> $$F(g_{n_r}^{-1}, g_{n_r} y_{n_r}) = F(e, y_{n_r})$$
>
> which is a contradiction. So done with the claim. □

Then it is a fact that different charts of this form differ by moving via the action of G. Using this, one can check that the transition maps for such an atlas as constructed above are smooth (as G is a Lie group). Then we are done. [**Exercise** to check these details.] □

2. Differential Forms and Curvature

2.1. Tensors.

We firstly give a brief linear algebra recap. We aim to study bilinear maps $U \times V \to W$, where U, V, W are vector spaces (note that these are not the same as elements of $\mathrm{Hom}(U \times V, W)$).

Definition 2.1. *Given vector spaces U, V, $\exists$ a vector space, denoted $U \otimes V$, called the* ***tensor product*** *of U, V, which comes with a bilinear map $\pi : U \times V \to U \otimes V$, and is uniquely characterised by the following* ***universal property****:*

> *"Given a vector space W and a bilinear map $\alpha : U \times V \to W$, then $\exists!$ linear map $\hat{\alpha} : U \otimes V \to W$ such that the diagram:*
>
> $$\begin{tikzcd} U \times V \arrow[r, "\alpha"] \arrow[d, "\pi"'] & W \\ U \otimes V \arrow[ur, "\hat{\alpha}"'] & \end{tikzcd}$$
>
> *commutes, i.e. tensor products allow us to extend bilinear maps on $U \times V$ to linear maps on $U \otimes V$."*

The existence of such a space $U \otimes V$ could be seen as a theorem itself, although we shall give several different constructions now. It is easy to check that the universal property characterises $U \otimes V$ when it exists.

Constructions of $U \otimes V$:

(a) Let $F(U \times V)$ to be the **free vector space** on elements of $U \times V$ (so, each element $(u, v) \in U \times V$ is seen as a basis element, which no interaction between different pairs). Then quotient out by the subspace X generated by the relations (i.e. these relations equal 0 in the quotient):
- $(u_1 + u_2, v) - (u_1, v) - (u_2, v)$
- $(u, v_1 + v_2) - (u, v_1) - (u, v_2)$
- $(au, v) - a(u, v)$
- $(u, av) - a(u, v)$

for $a \in \mathbb{R}$ (or $a \in \mathbb{F}$, the ground field). Then we declare $U \otimes V$ to be:
$$U \otimes V := F(U, V)/X$$
and one can check that this works.

(b) If $W = \mathbb{R}$, then the universal property says (by dualising):
$$(U \otimes V)^* = \mathrm{Bilinear}(U \times V, \mathbb{R})$$
(as these maps form a vector space, and because for every such map in one space we get one in the other), and so we can define $U \otimes V$ to be the unique space such that this holds.

(c) If $U = \langle u_1, \ldots, u_n \rangle$ and $V = \langle v_1, \ldots, v_m \rangle$ are finite dimensional, with bases given as here, then $U \otimes V$ is the vector space with basis: $\{u_i \otimes v_j\}_{ij}$, and so has dimension $\dim(U)\dim(V)$.

We can then set:
$$\left(\sum_i a_i u_i\right) \otimes \left(\sum_j b_j v_j\right) := \sum_{i,j} a_i b_j \cdot u_i \otimes v_j,$$
and then $\pi : U \times V \to U \otimes V$ is simply: $(u, v) \mapsto u \otimes v$. However, this construction only works in the finite dimensional cases.

From now on, we will assume that all of our vector spaces are finite dimensional (indeed, we are working with finite dimensional manifolds, and so the tangent spaces are finite dimesional vector spaces, so this is fine). So hence we can work with construction (c), which gives an explicit basis for the tensor product.

Some properties of $\otimes$ include:

(i) $U \otimes V \cong V \otimes U$ (i.e. commutative)

(ii) $(U \otimes V)^* \cong V^* \otimes U^*$

(iii) $\mathrm{Hom}(U, V) \cong U^* \otimes V$.

(iv) $\otimes$ is associative (but this requires more work to prove).

Recall: Given a vector bundle $E \to M$ and an operation on vector spcaes, such as dual or $\mathrm{Hom}(\cdot, \cdot)$, $\oplus$, etc, we can get a new vector bundle by taking vector space operations fibrewise,

e.g. if E is defined by a trivialising cover $\{U_\alpha\}_{\alpha \in A}$, with cocycles $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$, then:

- E^* is associated to the cocycles $\varphi_{\alpha\beta} := \psi^*_{\alpha\beta}$
- $E \times F$ is associated to the cocycle with matrix
$$\begin{bmatrix} \psi^E_{\alpha\beta} & 0 \\ 0 & \psi^F_{\alpha\beta} \end{bmatrix}$$
i.e. act on E parts via E's cocycles, and F by F's cocyles.

We therefore have vector bundles: $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q} \to M$ based on this tensor construction.

Definition 2.2. *A section of the vector bundle $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q}$ is called a* ***tensor of type (p, q)****.*

Example 2.1. *A $(0, q)$ tensor on a vector space V associates to each point an element of $(V^*)^{\otimes q}$. Using the properties of $\otimes$ given above, namely property (ii), we see that*
$$(V^*)^{\otimes q} \cong (V^{\otimes q})^* \cong \mathit{Bilinear}(\underbrace{V \times \cdots \times V}_{q \text{ times}}, \mathbb{R}).$$
So hence this is a map $T : V \times \cdots \times V \to \mathbb{R}$ such that it is linear in each component, i.e.
$$T(v_1, \ldots, v_j + a v'_j, v_{j+1}, \ldots, v_q) = T(v_1, \ldots, v_j, \ldots, v_q) + a T(v_1, \ldots, v'_j, \ldots, v_q).$$

So for example, the scalar product is a $(0,2)-tensor$, and the determinant of a matrix/linear map is a $(0,k)$-tensor, where $k = \dim(V)$.

Definition 2.3. *A $(0,q)$-tensor is called **alternating** if:*

$$T(v_1,\dots,v_i,\dots,v_j,\dots,v_q) = -T(v_1,\dots,v_j,\dots,v_i,\dots,v_q),$$

i.e. we get a change of signing when permuting two entries (i.e. antisymmetric).

There are two ways of viewing alternating tensors, and we shall use both viewpoints. We can either study alternating tensors as a subspace of all tensors (i.e. just take those that are alternating) or we can view them as their own space, as a quotient space of all tensors.

So for $\pi \in \mathrm{Sym}_q$ the symmetric group on q elements, let $(-1)^\pi := \mathrm{sign}(\pi)$. Then define the π'th permutation of T:

$$T^\pi(v_1,\dots,v_q) := T(v_{\pi(1)},\dots,v_{\pi(q)}).$$

So then clearly, we have

$$T \text{ is alternating} \iff T^\pi = (-1)^\pi T \quad \forall \pi.$$

So now note that from any $(0,q)$-tensor, we can generate an alternating tensor via the **alternating sum** of T:

$$\mathrm{Alt}(T) := \frac{1}{q!}\sum_{\pi\in\mathrm{Sym}_q}(-1)^\pi T^\pi.$$

Clearly this is an alternating tensor, and the $q! = |\mathrm{Sym}_q|$ factor is included as a normalisation factor, so that:

$$T \text{ is alternating} \iff \mathrm{Alt}(T) = T.$$

With this, we can also define a product of alternating tensors, called the **wedge product**, by:

$$\alpha\wedge\beta := \frac{(k+l)!}{k!l!}\mathrm{Alt}(\alpha\otimes\beta)$$

for α an alternating $(0,k)$-tensor and β an alternating $(0,l)$-tensor.

Remark: Here, we define $\alpha\otimes\beta$ by

$$(\alpha\otimes\beta)(v_1,\dots,v_{k+l}) := \alpha(v_1,\dots,v_k)\beta(v_{k+1},\dots,v_{k+l}).$$

With these products, the set of all $(0,q)$-tensors (with varying q) becomes an **algebra**, which we denote by

$$\bigoplus_{q\geq 0}(V^*)^{\otimes q}.$$

We then write $\Lambda^k V$ for the set of **alternating $(0,k)$-tensors**, and:

$$\Lambda^\star(V^*) := \bigoplus_{q\geq 0}\Lambda^q(V^*), \quad \text{where} \quad \Lambda^1(V^*) = V^*$$

with the wedge product, $\wedge$, for all alternating tensors.

Here, $\Lambda^0 V = \mathbb{R}$ ($= \mathbb{F}$, the ground field) (as $S_0 = \{e\}$ is trivial, and so just looking for constants), and $V^{\otimes 0} = \mathbb{R}$ (or $\mathbb{F}$).

Note: $1 \in \Lambda^0(V)$ is a unit for the wedge product, and note that clearly

$$\wedge : \Lambda^i V \otimes \Lambda^j V \to \Lambda^{i+j} V.$$

Lemma 2.1. *The wedge product $\wedge$ is associative.*

Proof. None given [**Exercise**].

□

Remark: If $T(V^*) = \oplus_{k \geq 0} (V^*)^{\otimes k}$ is the **tensor alegbra** of V^*, then we can define:

$$\Lambda^\star(V^*) := T(V^*)/I$$

where I is the multiplicative 2-sided ideal generated by elements of the form: $v \otimes v$, for $v \in V$. This is the alternative way of defining alternating tensors. In this set up, $\exists\ \pi : T(V^*) \to \Lambda^\star(V^*)$, and then:

$$a \wedge b := \pi(a \otimes b)$$

defines the wedge product (i.e. π just sends this to the alternating form for a, b which can be normalised to the wedge product).

Remark: The vector space $\Lambda^p V$ has dimension: $\binom{\dim(V)}{p}$. Indeed, if $I = \{i_1 < \cdots < i_p\}$ is a set of indices, and if we let $\varphi_I := \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}$, where the $\{\varphi_i\}_i$ form a basis of V, then the $\{\varphi_I\}_I$ forms a basis of $\Lambda^p(V)$. (This is because we know all p-wedge products must span, but by permuting them to make indices in increasing order, we get that we just need these. Note that $\alpha \wedge \beta = -\beta \wedge \alpha$ for $(0,1)$-tensors α, β. So hence if a wedge product contains two of the same $(0,1)$-tensor, it is zero.)

In particular, $\Lambda^{\dim(V)} V = \mathbb{R}$ (as it is 1-dimensional, spanned by $\varphi_1 \wedge \cdots \wedge \varphi_{\dim(V)}$), and $\Lambda^i V = \{0\}$ if $i > \dim(V)$ (as then each basis vector will contain two or more of a given φ_j, and so is zero).

So we see that $\Lambda^k V$ is always finite dimensional.

Note now that a map $\alpha : U \to V$ induces maps denoted $\Lambda^k \alpha$ on each $\Lambda^k U$, just by acting by α on each component, i.e. for each k,

$$\Lambda^k \alpha : \Lambda^k U \to \Lambda^k V \quad \text{via} \quad u_1 \wedge \cdots \wedge u_k \longmapsto \alpha(u_1) \wedge \cdots \wedge \alpha(u_k).$$

If α has matrix representation $(a_{ij})_{ij}$, then $\Lambda^k \alpha$ has matrix given by the $k \times k$ minors of $(a_{ij})_{ij}$, since

$$u_1 \wedge \cdots \wedge u_k \mapsto \alpha(u_1) \wedge \cdots \wedge \alpha(u_k) = \sum_{i_1, \ldots, i_k} a_{1i_1} \cdots a_{ki_k} \cdot u_{i_1} \wedge \cdots \wedge u_{i_k}$$

and then rearranging the wedge terms on the RHS, so that $i_1 < \cdots < i_k$, and modifying the front coefficient, we see that are are left with the expression for the determinant of each $k \times k$ block of $(a_{ij})_{ij}$, so done.

Also, we see in the case that $V = U$,

$$\Lambda^{\dim(U)}\alpha : \underbrace{\Lambda^{\dim(U)}U}_{\cong\mathbb{R}} \to \underbrace{\Lambda^{\dim(U)}U}_{\cong\mathbb{R}}$$

and this map must be given by (from the matrix expression above) $\mathbb{R} \to \mathbb{R}$, multiplication by $\det(\alpha)$.

So if $E \to M$ is a vector bundle of rank k, then each point of M has an associated k-dimensional vector space, and so we get the **determinant bundle**, $\det(E) := \Lambda^k E$, which is a canonical line bundle of M.

If E has cocycles $\psi_{\alpha\beta}$, then $\det(E)$ has cocycle matrices $\det(\psi_{\alpha\beta})$.

Definition 2.4. *Let M be a manifold. Then we define the vector space of* ***differential i-forms*** *on M to be:*

$$\Omega^i(M) := \Gamma\left(\Lambda^i(T^*M)\right)$$

i.e. each point $p \in M$ is associated to an alternating i-multilinear map on T_pM.

So in particular, we know that $\Omega^1(M) = \Gamma(\Lambda^1(T^*M)) = \Gamma(T^*M)$ are the differential 1-forms, as before. Note that also we have:

$$\Omega^i(M) \neq 0 \;\text{ if and only if }\; 0 \leq i \leq \dim(M)$$

$$\Omega^0(M) = C^\infty(M).$$

Then if $f : M \to N$ is smooth, we get the pullback f^* is a map of differential forms,

$$f^* : \Omega^*(N) \to \Omega^*(M)$$

as the dual map is $f^* : T^*N \to T^*M$, and then we can act on the spaces as described before (i.e. f^* acts on tensors by mapping each component).

The algebra $\Omega^\star(M) = \oplus_{i=0}^{\dim(M)}\Omega^i(M)$ has a product via $\wedge$, and then from how f^* acts (as it just acts on each component of a wedge product), we trivially have:

$$f^*(\omega \wedge \theta) = f^*(\omega) \wedge f^*(\theta).$$

Moreover, from how the dual operation acts on compositions, we know $(f \circ g)^*(\omega) = g^*(f^*(\omega))$.

So in local coordinates, if $f : U \to V$ with $U \subset \mathbb{R}^k$ has basis $\{x_1, \ldots, x_k\}$ and $V \subset \mathbb{R}^l$ has basis $\{y_1, \ldots, y_l\}$, and $\omega = \sum_I a_i \mathrm{d}y_I \in \Lambda^r(T^*V)$, where $\mathrm{d}y_I := \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_r}$, and $I = \{i_1 < \cdots < i_r\}$, then the pullback acts as:

$$f^*(\omega) = \sum_I a_I \mathrm{d}f_I,$$

where

$$\mathrm{d}f_I = \mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_r}$$

and

$$\mathrm{d}f_i = \sum_j \frac{\partial f_i}{\partial x_j}\mathrm{d}x_j.$$

This is because, f^* is linear, and so we can commute it with $\sum_I a_I$, and then

$$f^*(\mathrm{d}y_I) = f^*(\mathrm{d}y_{i_1}) \wedge \cdots \wedge f^*(\mathrm{d}y_{i_r})$$

and as $y_j \in C^\infty(V)$ is a coordinate function for each j, we know (since $\mathrm{d}y_{i_j} = (\partial/\partial y_{ij})^*$ is the dual of this derivation)

$$f^* \circ \mathrm{d}y_{i_j} = (y_{i_j}(f))^* = (f_{i_j})^* = \mathrm{d}f_{i_j}.$$

Here, f_i is the i'th component of f with respect to these coordinates.

Remark: Note that if $\alpha, \beta \in \Omega^1(M)$, then by skew-commutativity/as tensors in $\Omega^1(M)$ are alternating, we have $\alpha \wedge \beta = -\beta \wedge \alpha \in \Omega^2(M)$. By iterating this (writing in terms of a multi-wedge product of basis vectors and swapping), in general we see that

$$\boxed{\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha}$$

where $\alpha \in \Omega^{|\alpha|}(M)$ and $\beta \in \Omega^{|\beta|}(M)$. So this says that $\Omega^\star(M)$ is **graded commutative** (i.e. it depends on the ranks/gradings as to whether or not it is commutative or skew-commutative).

Remark: $\wedge$ is bilinear. [**Exercise** to check: this is just from how it is defined from the tensor product].

2.2. de Rham Cohomology.

So we know that $\Omega^\star(M)$ is a graded $\mathbb{R}$-algebra with respect to $\wedge$. Note that we already have:

$$\mathrm{d} : \underbrace{\Omega^0(M)}_{=C^\infty(M)} \to \underbrace{\Omega^1(M)}_{\Gamma(T^*M)}$$

via $f \mapsto \mathrm{d}f$, where $\mathrm{d}f(X) = X \cdot f$. Explicitly, in local coordinates $\{x_1, \dots, x_n\}$ on $U \subset M$, we have

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

This examples motivates finding more general d's, leading to:

Proposition 2.1 (Existence of Exterior Derivative). *$\exists$ an operator $\mathrm{d} : \Omega^i(M) \to \Omega^{i+1}(M)$, called the **exterior derivative**, such that:*

(i) d *is linear:* $\mathrm{d}(\omega_1 + \lambda\omega_2) = \mathrm{d}(\omega_1) + \lambda\mathrm{d}(\omega_2)$.

(ii) *Satisfies **Liebniz**, i.e.* $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}(\omega_1) \wedge \omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge \mathrm{d}(\omega_2)$.

(iii) *Satisfies **Poincaré**, i.e.* $\mathrm{d}(\mathrm{d}\omega) = 0$ *always, i.e.* $\mathrm{d}^2 = 0$ *in*

$$\Omega^i(M) \xrightarrow{\mathrm{d}} \Omega^{i+1}(M) \xrightarrow{\mathrm{d}} \Omega^{i+2}(M).$$

(iv) d *is natural, in the sense that if* $f : M \to N$, *then* $\mathrm{d} \circ f^* = f^* \circ \mathrm{d}$, *i.e. the diagram*

$$\begin{array}{ccc} \Omega^i(N) & \xrightarrow{\mathrm{d}} & \Omega^{i+1}(N) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega^i(M) & \xrightarrow[\mathrm{d}]{} & \Omega^{i+1}(M) \end{array}$$

commutes.

Proof. Later. □

Corollary 2.1. $(\Omega^\star(M), \mathrm{d})$ *forms a cochain complex, i.e.*

$$\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \cdots$$

is a sequence of $\mathbb{R}$*-vector spaces and linear maps such that any two consecutive maps compose to give 0.*

Proof. Immediate from the above proposition. □

Thus we can define the **cohomology** of this cochain complex:

Definition 2.5. *The **de Rham cohomology of M** is defined by:*

$$H^i_{dR}(M) := \frac{\ker(\mathrm{d} : \Omega^i(M) \to \Omega^{i+1}(M))}{Im(\mathrm{d} : \Omega^{i-1}(M) \to \Omega^i(M))},$$

where since $\mathrm{d}^2 = 0$*, this quotient makes sense.*

By construction, from the natural property (iv) of d, we see that

$$\mathrm{H}^\star_{\mathrm{dR}}(M) := \bigoplus_{i=0}^{\dim(M)} \mathrm{H}^i_{\mathrm{dR}}(M)$$

is invariant under diffeomorphisms of M, and so is a diffeomorphism invariant.

Notation: Elements of ker(d), i.e. ω such that $\mathrm{d}\omega = 0$ are called **closed forms**. Elements of Im(d), i.e. ω such that $\exists\theta$ such that $\omega = \mathrm{d}\theta$ are called **exact forms**.

Proof of Proposition 2.1. We have an intrinsic definition of $\mathrm{d} : \Omega^0 \to \Omega^1$, via $\mathrm{d}f(X) := X \cdot f$, as well as an expression in local coordinates, given before.

So we define:

$$\boxed{\mathrm{d}\left(\sum_I \omega_I \mathrm{d}x_I\right) := \sum_I \mathrm{d}\omega_I \wedge \mathrm{d}x_I},$$

where $\omega_I \in C^\infty(M)$ (on some $U \subset M$), and $\mathrm{d}x_I := \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, for $I = \{i_1 < \cdots < i_k\}$.

Claim: This is well-defined, i.e. independent of the choice of local coordinates.

Proof of Claim. To prove this, we first prove two subclaims. The point is that we have a coordinate-free definition on $\Omega^0(M)$. So if we can show that d only depends on this, then we are done.

> **Subclaim 1:** In the fixed coordinates, our definition satisfies properties (i)-(iii) of the proposition.

Proof of Subclaim 1. (i) and (ii) are immediate from the previous properties of $\wedge$. For (iii), note that by linearity of d it suffices to prove that $\mathrm{d}^2\omega = 0$ where $\omega = f\,\mathrm{d}x_I$. So,

$$\mathrm{d}(\mathrm{d}\omega) := \mathrm{d}\left(\sum_i \frac{\partial f}{\partial x_i}\cdot \mathrm{d}x_i \wedge \mathrm{d}x_I\right) := \sum_{i,j} \underbrace{\frac{\partial f^2}{\partial x_i \partial x_j}}_{\text{symmetric in } i,j} \cdot \overbrace{\mathrm{d}x_i \wedge \mathrm{d}x_j}^{\text{anti-symmetric in } i,j} \wedge \mathrm{d}x_I = 0.$$

So done with this subclaim.

□

Subclaim 2: If $\tilde{\mathrm{d}} : \Omega^k(U) \to \Omega^{k+1}(U)$ is any map satisfying (i)-(iii) of the proposition, and if $\mathrm{d} = \tilde{\mathrm{d}}$ on $\Omega^0(U)$, then $\mathrm{d} = \tilde{\mathrm{d}}$ on all forms.

Proof of Subclaim 2. We have:

$$\tilde{\mathrm{d}}(f\,\mathrm{d}x_I) = \tilde{\mathrm{d}}(f) \wedge \mathrm{d}x_I + f\,\tilde{\mathrm{d}}(\mathrm{d}x_I),$$

as $\tilde{\mathrm{d}}$ has the Leibniz property of d

$$= \mathrm{d}f \wedge \mathrm{d}x_I + f\,\tilde{\mathrm{d}}(\mathrm{d}x_I),$$

as $f \in \Omega^0(M)$ and $\mathrm{d} = \tilde{\mathrm{d}}$ on $\Omega_0(M)$. But $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, where the x_{i_j} are local coordinates functions, i.e. $x_{i_j} \in \Omega^0(M)$. So hence $\mathrm{d}x_{i_j} = \tilde{\mathrm{d}}x_{i_j}$, and so

$$\mathrm{d}x_I = \tilde{\mathrm{d}}x_{i_1} \wedge \cdots \wedge \tilde{\mathrm{d}}x_{i_k}.$$

Now we proceed by induction on k to show that $\tilde{\mathrm{d}}(\mathrm{d}x_I) = 0$. For $k = 1$, as $\mathrm{d}x_i = \tilde{\mathrm{d}}x_i$ and $\tilde{\mathrm{d}}^2 = 0$, this is trivially true.

Then for $k > 1$, we have

$$\begin{aligned}\tilde{\mathrm{d}}(\mathrm{d}x_I) &= \tilde{\mathrm{d}}\big((\mathrm{d}x_{i_1}) \wedge (\mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k})\big)\\ &= \underbrace{\tilde{\mathrm{d}}(\mathrm{d}x_{i_1})}_{=0 \text{ by } k=1 \text{ case}} \wedge (\mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k}) - \mathrm{d}x_{i_1} \wedge \underbrace{\big(\tilde{\mathrm{d}}(\mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k})\big)}_{=0 \text{ by induction on } k}.\end{aligned}$$

Hence by induction, we have $\tilde{\mathrm{d}}(\mathrm{d}x_I) = 0$ always. Hence:

$$\tilde{\mathrm{d}}(f\,\mathrm{d}x_I) = \mathrm{d}f \wedge \mathrm{d}x_I = \mathrm{d}(f\,\mathrm{d}x_I)$$

for all $f \in \Omega^0(M)$, $\mathrm{d}x_I \in \Omega^k(M)$. So by linearity, and elements of $\Omega^k(M)$ are linear combinations of such elements, are get $\mathrm{d} = \tilde{\mathrm{d}}$ on $\Omega^k(M)$, and so done. □

So hence to prove the claim, simply note that these two subclaims tell us that if there were two forms of d, then if they agree on $\Omega^0(M)$, then they agree everywhere/on all forms. Hence by taking the definition of d on $\Omega^0(M)$ as before, we get that this is well-defined.

□

So the claim has been proven. So all that remains to proven is then d as we defined it above does indeed satisfied property (iv).

Let $\omega = \varphi \cdot \mathrm{d}y_I \in \Omega^k(M)$ be a k-form on $V \subset N$ with local coordinates $y_1, \ldots, y_n$, where $\mathrm{d}y_I = \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_k}$. We shall prove (iv) by induction on k.

$k = 0$ case: We have for any $X \in \Gamma(TM) = \Omega^1(M)$,

$$(f^*(\mathrm{d}\varphi))(X) = \mathrm{d}\varphi(\mathrm{d}f(X)) = \mathrm{d}(\varphi \circ f)(X) = \mathrm{d}(f^*\varphi)(X)$$

where in the last line we have used the definition of the dual map on functions, i.e. $f^*(\varphi) = \varphi \circ f$. So as X was arbitrary, we have $f^* \circ \mathrm{d} = \mathrm{d} \circ f^*$ here.

$k > 0$ case: We have for ω as above (suffices to prove it for these ω by linearity)

$$\begin{aligned}
\mathrm{d}f^*(\omega) &= \mathrm{d}(f^*((\varphi \cdot \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_{k-1}}) \wedge \mathrm{d}y_{i_k})) \\
&= \mathrm{d}\big(f^*(\varphi \cdot \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_{k-1}})\big) \wedge f^*(\mathrm{d}y_{i_k}) \quad \text{by Leibniz, as } \mathrm{d}(f^*(\mathrm{d}y_{i_k})) = 0 \\
&= f^*(\mathrm{d}(\varphi \cdot \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_{k-1}})) \wedge f^*(\mathrm{d}y_{i_k}) \quad \text{by induction hypothesis} \\
&= f^*(\mathrm{d}\omega) \quad \text{using how we defined } f^* \text{ here, by acting on each component.}
\end{aligned}$$

So hence we are done proving (iv) by induction. So hence this definition above satisfied all properties, and so this shows existence, and so done.

□

Note: We proved uniqueness of d, provided we take the usual meaning of d on $\Omega^0(M) = C^\infty(M)$.

Now we proceed to prove some properties about the exterior derivative.

Lemma 2.2. *If $\omega \in \Omega^1(M)$, and $X, Y \in \Gamma(TM)$ are vector fields, then*

$$\mathrm{d}\omega(X,Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y]).$$

Proof. Note that both sides are linear in ω, and so it suffices to prove this for when $\omega = f\,\mathrm{d}g$ [**Exercise** to check this], for $f, g \in C^\infty(M)$.

In this case, $\mathrm{d}\omega = \mathrm{d}f \wedge \mathrm{d}g$, and so:

LHS: Here we have

$$\begin{aligned}
\mathrm{d}\omega(X,Y) &= (\mathrm{d}f \otimes \mathrm{d}g - \mathrm{d}g \otimes \mathrm{d}f)(X,Y) - \mathrm{d}f(X) \cdot \mathrm{d}g(Y) - \mathrm{d}g(X) \cdot \mathrm{d}f(Y) \\
&= (X \cdot f)(Y \cdot g) - (X \cdot g)(Y \cdot f).
\end{aligned}$$

RHS: And here we have

$$\begin{aligned}
&X \cdot (f\,\mathrm{d}g(Y)) - \dot{Y}(f\,\mathrm{d}g(X)) - f\,\mathrm{d}g([X,Y]) = X \cdot (f(Y \cdot g)) - Y \cdot (f(X \cdot g)) - f \cdot ([X,Y] \cdot g) \\
&= (X \cdot f)(Y \cdot g) + \underbrace{fX \cdot (Y \cdot g)}_{\text{cancels with (A)}} - (Y \cdot f)(X \cdot g) - \underbrace{f(Y \cdot (X \cdot g))}_{\text{cancels with (B)}} - f \cdot (\underbrace{X \cdot (Y \cdot g)}_{\text{(A)}} - \underbrace{Y \cdot (X \cdot g)}_{\text{(B)}}) \\
&= (X \cdot f)(Y \cdot g) - (Y \cdot f)(X \cdot g).
\end{aligned}$$

Here, we have used the definition of $\wedge$ in terms of tensor products, and how to evaluate the tensor product.

Hence both sides agree, so we are done.

□

Remarks:

(A) We can extend the result of Lemma 2.2: if $\omega \in \Omega^k(M)$ and $X_1,\dots,X_{k+1} \in \Gamma(TM)$ are vector fields, then in fact:

$$\mathrm{d}\omega(X_1,\dots,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}X_i \cdot \omega(X_1,\dots,\overbrace{\hat{X}_i}^{\text{omit}},\dots,X_{k+1}) + \sum_{1\leq i<j\leq k+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\underbrace{\hat{X}_i}_{\text{omit}},\dots,\underbrace{\hat{X}_j}_{\text{omit}},\dots,X_{k+1}).$$

Note that in particular this gives an intrinsic definition of the exterior derivative.

(B) Think about this lemma in the context of the Frobenius integrability theorem (we will return to this remark at some point).

Lemma 2.3 (The Poincaré Lemma). *If $M \cong \mathbb{R}^k$, or if M is any star-shaped open subset of $\mathbb{R}^k$, then:*

$$H^0_{dR}(M) \cong \mathbb{R} \quad \textit{and} \quad H^l_{dR}(M) = 0 \ \ \forall l \geq 1.$$

Proof. Clearly:

$$\begin{aligned}
\mathrm{H}^0_{\mathrm{dR}}(M) &= \ker\left(\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M)\right)\\
&= \left\{ f \in \overbrace{\Omega^0(M)}^{=C^\infty(M)} : \mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathrm{d}x_i = 0\right\}\\
&= \{f \in C^\infty(M) : \frac{\partial f}{\partial x_i} = 0 \ \forall i\}\\
&= \{\text{constant functions on } M\} \cong \mathbb{R},
\end{aligned}$$

since M is path connected, and so f takes some constant on all of M if its derivatives globally vanish, and so identify f by this constant.

[So in general, $\mathrm{H}^0_{\mathrm{dR}}(M) \cong \mathbb{R}^c$, where $c = \#$ of path components of M.]

For the second claim, the key observation is to construct $i : \Omega^l(M) \to \Omega^{l-1}(M)$ such that $i\circ \mathrm{d} + \mathrm{d}\circ i = \mathrm{id}_{\Omega^*(M)}$. Then given this, if ω is a closed form, then we have

$$\omega = \mathrm{id}(\omega) = (i\circ \mathrm{d} + \mathrm{d}\circ i)(\omega) = \mathrm{d}(i(\omega))$$

as $\mathrm{d}\omega = 0$, and so d is exact. So hence this shows that for $l \geq 1$, $\omega \in \ker(\mathrm{d}) \iff \omega \in \mathrm{Im}(\mathrm{d})$, and so $\mathrm{H}^l_{\mathrm{dR}}(M) = 0$, by definition of de Rham cohomology.

So we need to construct such an i. So if $\omega = \sum_I \omega_I \mathrm{d}x_I$, then define:

$$i(\omega)|_x := \sum_I \sum_{j=1}^l (-1)^{j-1}\left(\int_0^1 t^{l-1}\omega_I(tx)\mathrm{d}t\right)x_{i_j}\mathrm{d}x_{I\setminus\{i_j\}},$$

where $I = \{i_1 < \cdots < i_l\}$. Then explicit calculation shows that:

$$(\mathrm{d} \circ i + i \circ \mathrm{d})(\omega)|_x = \sum_I \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \left(t^l \omega_I(xt) \right) \right) \mathrm{d}x_I = \sum_I ((1 \cdot \omega_I(x) - 0))\, \mathrm{d}x_I = \omega|_x,$$

which completes the proof.

□

The Poincaré Lemma tells us that locally on any manifold, all closed l-forms are exact, for all $l \geq 1$. This is because locally a manifold is isomorphic to $\mathbb{R}^k$ for some k, and so we can apply the Poincaré Lemma locally.

Remark: The i defined in the proof of the Poincaré Lemma may look kind of bizarre. It is called the **contracting homotopy**, and is very closely related to the interior product (see Definition 2.10).

Note: From the definition of $\mathrm{H}^\star_{\mathrm{dR}}$, we see that each group is a subquotient of infinite dimensional vector spaces.

So our next goal is to show for 'decent' M, $\mathrm{H}^\star_{\mathrm{dR}}(M)$ is in fact finite dimensional.

Fact: It turns out that

$$\mathrm{H}^\star_{\mathrm{dR}}(M) \equiv \mathrm{H}^\star_{\mathrm{sing}}(M; \mathbb{R})$$

are actually naturally isomorphic, i.e. de Rham cohomology is "the same" as the singular cohomology of M (with coefficients in $\mathbb{R}$).

2.3. **Orientation and Integration.**

Recall: If V is an n-dimensional vector space, then $\Lambda^n V \cong \mathbb{R}$. In the case of a tangent bundle, $\Lambda^n(T^*M^n)$ is called the determinant line bundle of M (but $\dim(T^*M) \neq \dim(M)$ in general, i.e. if E is a vector bundle with cocycle transition matrices $\psi_{\alpha\beta}$, then $\det(E) := \Lambda^{\mathrm{rank}(E)}E$ had transition matrices $\det(\psi_{\alpha\beta})$).

Concretely, if $f : U \to V$ with $U, V \subset \mathbb{R}^n$ are open, with f smooth, then:

$$f^*(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n) = \det(\mathrm{D}f)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

is the transformation, since:

$$\begin{aligned} f^*(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n) &= f^*(\mathrm{d}y_1) \wedge \cdots f^*(\mathrm{d}y_n) \\ &= \mathrm{d}(f^* y_1) \wedge \cdots \mathrm{d}(f^* y_n) \\ &= \mathrm{d}(y_1 \circ f) \wedge \cdots \wedge \mathrm{d}(y_n \circ f) \\ &= \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \end{aligned}$$

and as $\mathrm{d}f_i = \frac{\partial f_i}{\partial x_j}\mathrm{d}x_j$ (sum over j) we get the above.

Definition 2.6. *We say that an n-manifold M is **orientable** if it admits a nowhere zero n-form, $\omega \in \Omega^n(M)$. Such an n-form is called a **volume form**.*

Lemma 2.4. *An n-manifold is orientable $\iff$ It admits an atlas $(U_\alpha, \varphi_\alpha)_\alpha$ such that all transition maps have > 0 determinant.*

[This is the intuitive definition of orientable, as we do not want tangents to flip.]

Remark: We say that a diffeomorphism $f : U \to V$ with $U, V \subset \mathbb{R}^n$ open is **orientation preserving** if $\det(Df|_x) > 0\ \forall x \in U$.

Proof. $(\Rightarrow)$: Given a nowhere zero n-form ω, we can consider the atlas A of all charts $(U_\alpha, \varphi_\alpha)$ on M such that if $x_1, \ldots, x_n$ are the local coordinates induced by $(U_\alpha, \varphi_\alpha)$, we have $\omega\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) > 0$.

[If $(U_\alpha, \varphi_\alpha)$ is <u>any</u> chart on M, then composing φ_α with a reflection of $\mathbb{R}^n$ if necessary, then we get a <u>new</u> chart $(U_\alpha, \varphi'_\alpha) \in A$ (as this will flip the sign of ω), So the charts of A do indeed cover M and so yield an atlas, i.e. we just look at which basis on the tangent space is positively/negatively orientated with respect to ω.]

Then if we look at ω in any chart on A, we get:

$$\omega = f \cdot \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$$

for some $f > 0$, since $0 < \omega\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right) = f$. So hence A is an atlas of orientation preserving charts.

$(\Leftarrow)$: Conversely, if $(U_\alpha, \varphi_\alpha)_\alpha$ is an atlas such that the transition maps are orientation preserving, and $\{\rho_\alpha\}_\alpha$, $\rho_\alpha : U_\alpha \to \mathbb{R}_{\geq 0}$ is a partition of unity subordinate to this atlas, then if $\tilde{\omega}_\alpha = \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ is the standard n-form on $\varphi_\alpha(U_\alpha)$, then we see that for $(\varphi_\beta \circ \varphi_\alpha^{-1})(\tilde{\omega}_\alpha)$ on $\varphi_\beta(U_\beta)$, we get a positive multiple of the corresponding n-form $\tilde{\omega}_\beta$.

In particular, if $\omega_\alpha \in \Omega^n(U_\alpha)$ is the n-form corresponding to $\tilde{\omega}_\alpha$, then $\sum_\alpha \rho_\alpha \omega_\alpha$ is nowhere zero, since pointwise it is a positive linear combination of copies of the same form (as each point will lie in some U_α). So hence we have found our form and so we are done.

□

Remark: Since $\Lambda^n(T^*M)$ is a real line bundle, the existence of a volume form is also equivalent to asking that this bundle is globally trivial (and not just locally), i.e.

$$\Lambda^n T^*M \cong M \times \mathbb{R}$$

are isomorphic as bundles.

An **orientation of M** is then just a choice of volume form, up to the reparameterisation by everywhere positive functions.

Lemma 2.5. *If M is an n-manifold with boundary ∂M, then an orientation of M canonically determines an orientation on ∂M, called the* ***Stoke's orientation****.*

Proof. If $\varphi : U \hookrightarrow \mathbb{H}^n = \{x_1 \geq 0\}$ is a chart at $p \in \partial M$, then:

$$\mathrm{d}\varphi|_p : T_pM \xrightarrow{\cong} \mathbb{R}^n,$$

and $T_p(\partial M) \subset T_pM$ is a codimension 1 subspace.

Now pick a complement $E_M \subset T_pM$ to $T_p(\partial M)$ (i.e. the direction we are missing). Then, $\mathrm{d}\varphi|_p(E_M)$ has positive or negative x_1-component (corresponding to whether it is an inward or outward tangent at ∂M).

So orient ∂M by declaring that a basis of $T_p(\partial M)$, $\{e_1, \ldots, e_{n-1}\}$, is positively oriented $\Longleftrightarrow (n, e_1, \ldots, e_{n-1})$ is a positively orientated basis for T_pM, where $n =$ the outward normal to ∂M (i.e. if $\omega(n, e_1, \ldots, e_{n-1}) > 0$ for ω our volume form on M).

This then works, and so done.

□

Remark: This is called the Stoke's orientation because it will make Stoke's theorem work by giving the correct sign. We shall see this in the proof later.

Now if $U, V \subset \mathbb{R}^n$ are open, and $f : U \to V$ is a diffeomorphism, then:

$$\int_V a(y)\mathrm{d}y_1 \cdots \mathrm{d}y_n = \int_U (a \circ f)(x) \left|\det(\mathrm{d}f|_x)\right| \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

using the usual 'change of variables' formula in $\mathbb{R}^n$.

This shows that the value of "$\int_V a$", is **not intrinsic**, as it depends on the choice of coordinates, and in particular, the orientation (via $|\det(\mathrm{d}f|_x)|$).

But if $|\det(\mathrm{d}f|_x)| = \det(\mathrm{d}f|_x)$ everywhere on U, then the integral of $a\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ **is intrinsic**, i.e. to make this intrinsic, we need an orientation. Plus we should talk about integrating differential forms/volume forms, which are intrinsic, and not integration scalars/functions.

So this motivates the integration of differential forms (since it is intrinsic).

Definition 2.7. *Let $\Omega^\star_{ct}(M)$ be the subspace of $\Omega^\star(M)$ consisting of the* ***compactly supported differential forms****.*

So if M is compact that $\Omega^\star_{\mathrm{ct}}(M) = \Omega^\star(M)$ and this will mean that we can integrate any form on M.

Note that since $\mathrm{d} : \Omega^i(M) \to \Omega^{i+1}(M)$ is **local** as an operator, we have:

$$(\Omega^\star_{\mathrm{ct}}(M), \mathrm{d}) \subset (\Omega^\star(M), \mathrm{d}) \quad \text{is a subcomplex.}$$

Locality here is very powerful, and what makes de Rham cohomology different to usual cohomologies of compactly supported objects.

Remark: $\Omega^\star_{\mathrm{ct}}(M)$ has different functorial properties: a general smooth map of open sets **does not** induce a pullback on $\Omega^\star_{\mathrm{ct}}$ (as compact support in image $\not\Rightarrow$ compact support on domain).

But if $i : U \hookrightarrow V$, $U \subset V$, is the inclusion of an open set, and $\alpha \in \Omega^i_{\mathrm{ct}}(U)$, then α does have a pushforward under i, $i_*(\alpha) \in \Omega^i_{\mathrm{ct}}(V)$, where

$$i_*(\alpha) = \text{extension of } \alpha \text{ by } 0 \text{ in } V \backslash U,$$

i.e. as α must be zero at ∂U, we can extend α to just be 0 on $V \backslash U$ and we get an element of $\Omega^i_{\mathrm{ct}}(V)$. So we can push forward by inclusions.

Proposition 2.2 (Integration of Differential Forms). *If M is an oriented $\underline{n}$-manifold, then $\exists$ a well-defined linear map on compact supported n-forms:*

$$\int_M : \Omega^n_{ct}(M) \to \mathbb{R}$$

*which is called **integration on M** (defined currently for n-forms).*

Proof. Take an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of charts for M and a partition of unity subordinate to this cover, $\rho_\alpha : U_\alpha \to \mathbb{R}_{\geq 0}$.

Then for $\omega \in \Omega^n_{\mathrm{ct}}(M)$, define:

$$\boxed{\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \cdot \omega}$$

where here, $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$, and the RHS is a sum of classical multivariable integrals on $\mathbb{R}^n$, i.e.

$$\int_{U_\alpha} \alpha = \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*(\alpha)$$

where the RHS here is the integral of the pullback of α to $\mathbb{R}^n$ (as $\left(\varphi_\alpha^{-1}\right)^*$ is $\mathbb{R}^n$ valued).

[Note how this expression makes sense, as if we could change each integral to one over M, since ρ_α is 0 outside of U_α, then since $\sum_\alpha \rho_\alpha = 1$ on M, this would just give $\int_M \omega$.]

Then since $\omega \in \Omega^n_{\mathrm{ct}}(M)$, we know $\mathrm{supp}(\omega)$ is compact, and so wlog the sum on the RHS is finite (as we only need to cover $\mathrm{supp}(\omega) \subset M$ for the integral, which we can do by the atlas and then restrict to a finite cover by compactness. Then take a partition of unity subordinate to this finite cover, etc, as above).

Claim: This definition is well-defined, i.e. is independent of the choice of cover $\{U_\alpha\}_\alpha$ and the choice of partition of unity.

Proof of Claim. If $\{(V_j, \psi_j)\}_{j\in B}$ is another cover with partition of unity $\eta_j : V_j \to \mathbb{R}_{\geq 0}$, then:

$$\int_{U_i} \rho_i \eta_j \omega = \int_{V_j} \rho_i \eta_j \omega,$$

as $\text{supp}(\rho_i \eta_j) \subset U_i \cap V_j$, and hence this is a Euclidean change of variables for n-forms under orientation preserving charts (i.e. both can be written as an integral over $U_i \cap V_j$. Then change coordinates from those on U_i to those on V_j in the set $U_i \cap V_j$. But then as the transformation is invertible, and is orientation preserving, the determinant factor is simply $+1$, so nothing changes).

So hence:

$$\sum_i \int_{U_i} \rho_i \omega = \sum_i \int_{U_i} \rho_i \underbrace{\left(\sum_j \eta_j\right)}_{\equiv 1} \omega = \sum_{i,j} \int_{U_i} \rho_i \eta_j \omega$$

where we have used the fact that from our definition, $\int$ is linear, and so we can exchange it with finite sums (which we wlog have, from the above).

But then by the same argument/symmetry, we also have

$$\sum_j \int_{V_j} \eta_j \omega = \sum_{i,j} \int_{V_j} \rho_i \eta_j \omega.$$

So hence using the equality above relating the integrals over U_i and V_j, we get that

$$\sum_i \int_{U_i} \rho_i \omega = \sum_j \int_{V_j} \eta_j \omega,$$

and so hence $\int_M \omega$ is well-defined. □

So hence we are done.

□

Note that clearly $\int_M : \Omega^n_{\text{ct}}(M) \to \mathbb{R}$ is non-trivial (i.e. not identically zero - this is seen by taking a form which is supported in the ball, say, which is > 0, e.g. the standard form with smooth cutoff), we see that it is **surjective** (as it is linear onto $\mathbb{R}$ and non-trivial, and so we get all of $\mathbb{R}$ by scaling, as scaling an element of $\Omega^n_{\text{ct}}(M)$ does not leave it, i.e. ω of compact support $\Rightarrow c\omega$ has compact support for all $c \in \mathbb{R}$).

Now we prove the big generalisation of the divergence theorem, Stoke's theorem, etc, from usual multivariable calculus:

Theorem 2.1 (Stoke's Theorem)**.** *If M is an oriented n-manifold with boundary ∂M and $\omega \in \Omega^{n-1}_{ct}(M)$, then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Note: ∂M is closed in M, and so we have:

$$\omega|_{\partial M} = i^*\omega,$$

where $i : \partial M \to M$ is the inclusion. Hence this is something with compact support (as ω has compact support) and so we can define $\int_{\partial M} \omega$ as before.

Remark: In particular, Stoke's theorem tells us that if M is a closed n-manifold, i.e. compact without boundary (recall that the boundary of M is $\partial M := \overline{M}\backslash M$, which is empty if M is closed), then we have for all such ω,

$$\int_M \mathrm{d}\omega = 0.$$

Corollary 2.2. *If M is a closed n-manifold, then $\int_M : \Omega^n(M) \to \mathbb{R}$ descends to a (non-trivial) linear map $\int_M : H^n_{\mathrm{dR}}(M) \to \mathbb{R}$, i.e. it only depends on the de Rham equivalence class.*

Proof of Corollary. Suppose $[\omega_1] = [\omega_2] \in \mathrm{H}^n_{\mathrm{dR}}(M)$. Then we need to show that $\int_M \omega_1 = \int_M \omega_2$. Since integration is linear and $[\omega_1 - \omega_2] = 0 \in \mathrm{H}^n_{\mathrm{dR}}(M)$, it suffices to prove that if $[\omega] = 0 \in \mathrm{H}^n_{\mathrm{dR}}(M)$, then $\int_M \omega = 0$. So in this case, by definition of de Rham cohomology, $\exists\ \varphi \in \Omega^{n-1}(M)$ with $\omega = \mathrm{d}\varphi$. So by Stoke's theorem,

$$\int_M \omega = \int_M \mathrm{d}\varphi = 0$$

by the above remark. So done.

□

Proof of Stoke's Theorem. We first consider the basic case of when $\partial M = \emptyset$.

Then we have a manifold without boundary (and so all charts are to open subsets of $\mathbb{R}^n$). So let $\omega \in \Omega^{n-1}_{\mathrm{ct}}(M)$, and write $\omega = \sum_i \rho_i \omega$, where as usual, $(\rho_i)_i$ is a partition of unity subordinate to the cover $\{(U_i, \varphi_i)\}_i$ (of M or $\mathrm{supp}(\omega)$).

Then on U_i, we can write (as we know a basis of $\Omega^{n-1}(M)$):

$$\rho_i\omega = a_1 \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n + a_2 \mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \cdots \wedge \mathrm{d}x_n + a_n \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1},$$

where $a_i \in C^\infty_{\mathrm{ct}}(U_i)$. So hence [**Exercise** to check]:

$$\mathrm{d}(\rho_i\omega) = \left(\frac{\partial a_1}{\partial x_1} - \frac{\partial a_2}{\partial x_2} + \cdots + (-1)^{n-1}\frac{\partial a_n}{\partial x_n}\right) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

via the commutative properties of $\wedge$, and since $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$ (i.e. if we have two of the same terms in a wedge product it is zero, so these terms are the only ones which survive).

So now let $\varphi_i : U_i \to \mathbb{R}^n$. Then we know (from our definition of integration of differential forms)

$$\int_{U_i} \frac{\partial a_1}{\partial x_1} \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \frac{\partial a_1}{\partial x_1}\, \mathrm{d}x_1\right) \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

where we have used Fubini's theorem, and decomposed the integral over integrals over $\mathbb{R}$ for each term.

But:

$$\int_{\mathbb{R}} \frac{\partial a_1}{\partial x_1}\,\mathrm{d}x_1 = \lim_{N\to\infty}[a_1]_{-N}^{N} = 0$$

since $\mathrm{supp}(a_1) \subset [-N,N]\times\mathbb{R}^{n-1}_{x_2,\dots,x_n}$ for some N. So hence we see

$$\int_{U_i} \frac{\partial a_1}{\partial x_1}\,\mathrm{d}x_2\wedge\cdots\wedge\mathrm{d}x_n = 0$$

and similarly the other terms in the expression for $\mathrm{d}(\rho_i\omega)$ give 0 when integrated.

So hence $\int_{U_i}\mathrm{d}(\rho_i\omega) = 0$ for each i, and so hence if M has no boundary, we have by linearity of d,

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}(\sum_i \rho_i\omega) = \sum_i \int_M \mathrm{d}(\rho_i\omega) = 0.$$

So hence we have proven the simple case, i.e. when $\partial M = \emptyset$.

So now suppose $\partial M \neq \emptyset$. Then again write $\omega = \sum_i \rho_i\omega$, for $\{\rho_i\}_i$ again a partition of unity subordinate to an atlas $\{(U_i,\varphi_i)\}_i$ on M. Then on U_i, we have the same expressions for $\rho_i\omega$ and $\mathrm{d}(\rho_i\omega)$ as above.

Then if the chart $U_i \subset M\backslash\partial M$ (i.e. is away from the boundary), then by the same calculation as above, Fubini's theorem gives $\int_{U_i}\mathrm{d}(\rho_i\omega) = 0$.

So it suffices to just consider the case when $U_i\cap\partial M \neq \emptyset$. So say $U_i = \{x_1 \geq 0\}\cap B^n(1) \subset M$, "centred" at a point of ∂M (wlog we can find such coordiantes). Then:

$$\int_{U_i}\mathrm{d}(\rho_i\omega) = \int_{\mathbb{R}^n\cap\{x_1\geq 0\}}\left(\frac{\partial a_1}{\partial x_1}+\cdots+(-1)^{n-1}\frac{\partial a_n}{\partial x_n}\right)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n.$$

So every integral is the same as before, except for the one over x_1, since now we have only integrating over $[0,\infty)$ instead of $(-\infty,\infty)$. So hence all the integrals involving $\frac{\partial a_i}{\partial x_i}$ for $i > 1$ vanish just as before. Hence this equals

$$= \int_{\mathbb{R}^{n-1}}\left(\int_0^\infty \frac{\partial a_1}{\partial x_1}\,\mathrm{d}x_1\right)\mathrm{d}x_2\cdots\mathrm{d}x_n = \int_{\mathbb{R}^{n-1}}[a_1]_0^\infty\,\mathrm{d}x_2\cdots\mathrm{d}x_n$$

$$= -\int_{\mathbb{R}^{n-1}} a_1(0,x_2,\dots,x_n)\,\mathrm{d}x_2,\cdots\mathrm{d}x_n = \int_{U_i\cap\partial M}\rho_i\omega,$$

where we have used the fact that $a_1 = 0$ outside of some compact set (as compact support) and so when evaluating the x_1 integral we are just left with the value at $x_1 = 0$, and then since from before, $(\rho_i\omega)|_{\partial M} = a_1(0,x_2,\dots,x_n)\,\mathrm{d}x_2\cdots\mathrm{d}x_n$. But then note that we are orienting the boundary with respect to an outwards normal vector, i.e. if $e_2,\dots,e_n$ is an oriented basis for $\mathbb{R}^{n-1} = T_p(\partial M)$ for $p\in\partial M$, then $(-e_1,e_2,\dots,e_n)$ is an oriented basis for T_pM (this deals with the minus sign - this is the point of the Stoke's orientation). So hence:

$$\int_{U_i} \mathrm{d}(\rho_i \omega) = \int_{U_i \cap \partial M} \rho_i \omega$$

for all i. So summing this over i and using that the U_i cover M and $\sum_i \rho_i \equiv 1$ on M, we get the result.

□

Recall: We now have that $\int_M : \Omega^n_{\mathrm{ct}}(M) \to \mathbb{R}$ descends to a map $\mathrm{H}^n_{\mathrm{ct}}(M) \to \mathbb{R}$, which is non-zero when M is orientable, provided $\partial M = \emptyset$.

Theorem 2.2. *If M is connected and orientable, then (after choosing an orientation) we have a distinguished isomorphism:*

$$H^n_{ct}(M) \xrightarrow{\cong} \mathbb{R} \quad \textit{via} \quad \int_M .$$

Proof. We choose an n-form $\omega \in \Omega^n_{\mathrm{ct}}(M)$ such that

$$\mathrm{supp}(\omega) \subset \overline{B^n(1)} \subset \underbrace{U}_{\cong \mathbb{R}^n} \subset M \quad \text{which has} \quad \int_M \omega = 1.$$

(this will allow us to reduce to the $\mathbb{R}^n$ case, which is the key lemma below). We need to know that any other compactly supported n-form on M differs from a multiple of ω be an <u>exact</u> compactly supported n-form. We now need the following key lemma:

> **Key Lemma:** Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and compactly supported in $(-1,1)^n$. Suppose also that $\int_{\mathbb{R}^n} f \; \mathrm{d}x_1 \cdots \mathrm{d}x_n = 0$. Then, $\exists$ smooth $u_i : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{supp}(u_i) \subset (-1,1)^n$ such that
>
> $$f = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} \quad (= \nabla \cdot u)$$
>
> i.e. $f \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is exact [i.e. if
>
> $$\eta = u_1 \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n - u_2 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n + \cdots + (-1)^{n-1} u_n \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1},$$
>
> then $f \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \mathrm{d}\eta$ (similar calculation to that in the proof of Stoke's Theorem).]

With this Key Lemma, we see that if $\omega \in \Omega^n_{\mathrm{ct}}(\mathbb{R}^n)$, then

$$\omega = \mathrm{d}\eta \text{ for some } \eta \in \Omega^{n-1}_{\mathrm{ct}}(\mathbb{R}^n) \iff \int_{\mathbb{R}^n} \omega = 0.$$

The $(\Longrightarrow)$ direction is just by Stoke's Theorem. The $(\Longleftarrow)$ implication is the Key Lemma, as we shall see.

Proof of Key Lemma. Consider first the $n=1$ case. Given $g:\mathbb{R}\to\mathbb{R}$ with compact support, we seek f such that $g=\frac{\mathrm{d}f}{\mathrm{d}x}$. Then we can let:

$$f(x)=\int_{-\infty}^{x} g(t)\mathrm{d}t.$$

Then this f will have compact support precisely when $\int_{\mathbb{R}} g\mathrm{d}t=0$ (as g has compact support, so integral won't change after some point, etc), and so with the integral assumption of $\int_{\mathbb{R}} g\mathrm{d}t=0$ this all works. So we have proven the Key Lemma when $n=1$.

In many variable, we do something similar. In general, pick $\rho:\mathbb{R}\to[0,1]$ smooth such that:

$$\rho(t)=\begin{cases}0 & \text{if } t\le -1+\varepsilon\\ 1 & \text{if } t\ge 1-\varepsilon\end{cases}$$

for some small $\varepsilon>0$. Note that $\rho'(t)$ has support in $(-1+\varepsilon,1-\varepsilon)$. Then inductively define functions f_i as follows (for $0\le i\le n$):

$$f_n=f \quad \text{(given)}$$

$$f_i(x)=\int_{-1}^{1}\cdots\int_{-1}^{1} f(x_1,\dots,x_i,\xi_{i+1},\dots,\xi_n)\rho'(x_{i+1})\cdots\rho'(x_n)\mathrm{d}\xi_{i+1}\cdots\mathrm{d}\xi_n.$$

So first note that $f_0(x)\equiv 0$, as:

$$\begin{aligned} f_0(x)&=\int_{-1}^{1}\cdots\int_{-1}^{1} f(\xi_1,\dots,\xi_n)\rho'(x_1)\cdots\rho'(x_n)\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n\\ &=\rho'(x_1)\cdots\rho'(x_n)\underbrace{\int_{\mathbb{R}^n} f(\xi)\mathrm{d}\xi}_{=0 \text{ by assumption}}=0. \end{aligned}$$

We then want:

$$u_i(x)=\int_{-1}^{x_i}(f_i-f_{i-1})(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\,\mathrm{d}t.$$

Then as the f_i are supported in $(-1,1)^n \Rightarrow$ the u_i have support in $(-1,1)^n$. Also, as

$$\left.\frac{\partial u_i}{\partial x_i}\right|_x=f_i(x)-f_{i-1}(x),$$

we see that

$$\sum_i\frac{\partial u_i}{\partial x_i}=\sum_i(f_i-f_{i-1})=f_n-f_0=f.$$

So we are done with the proof of this lemma. □

So given this lemma, if $\tilde{\omega}$ is another compactly supported n-form (with support contained in V, say) then as usual we write:

$$\tilde{\omega}=\sum_i\varphi_i\tilde{\omega}$$

for $\{\varphi_i\}_i$ a partition of unity. So, hence $\mathrm{supp}(\varphi\tilde{\omega})\subset B^n(1)\subset U_i$, and, using the compact support and the fact that all these partition of unity sums are finite sums, we see that it is sufficient to prove that $\exists\ c_i\in\mathbb{R}$ and $\eta_i\in\Omega^{n-1}_{\mathrm{ct}}(M)$ such that

$$\varphi_i\tilde{\omega}=c_i\omega+\mathrm{d}\eta_i.$$

We can then find a chain of discs $U_0, U_1, \ldots, U_N$ with $U_0 = U$ and $U_N = V$ such that $U_i \cap U_{i+1}$ is connected and such that the transition maps between neighbouring discs have positive determinant (as M is oriented).

Now we pick ω_i compactly supported in $U_i \cap U_{i+1}$ such that $\int_{U_i} \omega_i > 0$. So as $U_i \cap U_{i+1} \neq \emptyset$ and $\mathrm{supp}(\omega_i) \subset U_i \cap U_{i+1}$, when integrating ω_i and ω_{i+1} over U_{i+1} we get a non-zero contribution from each of them. So we can choose $c_i \neq 0$ such that

$$\int_{U_i} \omega_{i+1} - c_i \omega_i = 0.$$

Then the Key Lemma implies that $\omega_{i+1} - \tau_i \omega_i = \mathrm{d}\eta_i$ for some τ_i. Moreover this is true for each i (with the η_i compactly supported in U_i), since the lemma tells us forms of integral 0 in a ball are exact with compactly supported primitive (i.e. if $\mathrm{d}\eta = \omega$, then η is a **primitive** of ω). So hence we have:

$$\begin{aligned}
\omega_1 - c_0 \omega &= \mathrm{d}\eta_0 \\
\omega_2 - c_1 \omega_1 &= \mathrm{d}\eta_1 \\
&\vdots \\
\tilde{\omega} - c_{N-1}\omega_{N-1} &= \mathrm{d}\eta_{N-1}.
\end{aligned}$$

and so adding we get:

$$\tilde{\omega} = c_0 \cdots c_{n-1} \omega + \mathrm{d}(c_1 \eta_0 + \cdots + c_{n-1}\eta_{n-2} + \eta_{n-1})$$

i.e.

$$[\tilde{\omega}] = c[\omega] \in \mathrm{H}^n_{\mathrm{ct}}(M),$$

for some c, i.e. it is unique up to a factor of $\mathbb{R}$, as required.

$\square$

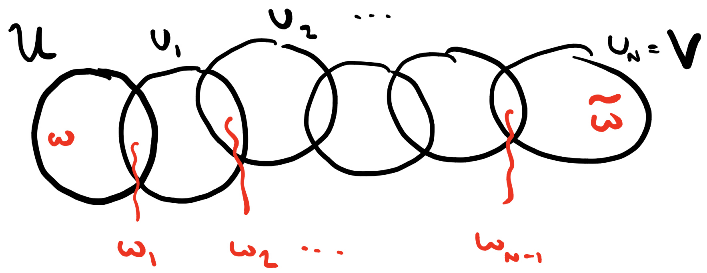

FIGURE 1. An illustration of the sets U_i and ω_i used in the proof of Theorem 2.2.

Exercise: Show that if M is connected **but not** orientable, then $\mathrm{H}^n_{\mathrm{ct}}(M) = \{0\}$, i.e. we have $\mathrm{d}\omega = 0 \iff \omega = \mathrm{d}\eta$ for some η.

So hence we see that $\mathrm{H}^n_{\mathrm{ct}}$ being non-trivial gives another equivalent characterisation of orientability (which we can use for more general situations).

So in summary, we have now seen, if M is a manifold with dimension n:

- $\mathrm{H}^0_{\mathrm{dR}}(M) \cong \mathbb{R}$ ($\equiv$ constant functions) if M is connected

- $\mathrm{H}^n_{\mathrm{ct}}(M) \cong \mathbb{R}$, and so hence $\mathrm{H}^n_{\mathrm{dR}} \cong \mathbb{R}$ (as this is 1-dimensional and $\mathrm{H}^n_{\mathrm{ct}}(M)$ is a subspace), if M is connected, compact and orientable.

2.4. **Manifold Type.**

Definition 2.8. *We say that a manifold M has **type k** if it admits a covering by k open sets U_i such that $U_i \cong \mathbb{R}^n$ for all i, and for all $I = \{i_1 < \cdots < i_k\}$, writing $U_I := U_{i_1} \cap \cdots \cap U_{i_k}$, then we either have $U_I \cong \mathbb{R}^n$ (i.e. a C^∞ diffeomorphism to $B_1(0) \cong \mathbb{R}^n$), or $U_I = \emptyset$.*

*We say that M has **finite type** if it has type k for some $k \geq 1$.*

Example: S^2 has type 2, and so is of finite type.

Later, we will show:

Proposition 2.3. *If M is a smooth manifold which is closed, or is the interior of a compact manifold with boundary, then M has finite type.*

Proof. See later [Sketch proof: Small balls in M are geodesically convex with respect to any chosen Riemannian metric].

□

Now a quick recap of some basis (co)homology theory.

Definition 2.9. *Recall that a **cochain complex** $(C^\bullet, \mathrm{d})$ was a sequence of vector spaces $(C^i)_i$ such that*

$$\cdots \xrightarrow{\mathrm{d}} C^{i-1} \xrightarrow{\mathrm{d}} C^i \xrightarrow{\mathrm{d}} C^{i+1} \xrightarrow{\mathrm{d}} \cdots$$

has $\mathrm{d} \circ \mathrm{d} = 0$.

*With this, the **cohomology of $(C^\bullet, \mathrm{d})$** is:*

$$H^i(C^\bullet, \mathrm{d}) := \frac{\ker(\mathrm{d} : C^i \to C^{i+1})}{Im(\mathrm{d} : C^{i-1} \to C^i)}.$$

*We seay that $(C^\bullet, \mathrm{d})$ is **exact** if it has trivial cohomology, i.e. at each stage C^i, we have $\ker(\mathrm{d} : C^i \to C^{i+1}) = Im(\mathrm{d} : C^{i-1} \to C^i)$.*

*Then a **short exact sequence (s.e.s)** of cochain complexes $0 \to C^\bullet \xrightarrow{\alpha} D^\bullet \xrightarrow{\beta} E^\bullet \to 0$ is a diagram:*

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\mathrm{d}"] & \vdots \arrow[d, "\mathrm{d}"] & \vdots \arrow[d, "\mathrm{d}"] & \\
0 \arrow[r] & C^i \arrow[r, "\alpha"] \arrow[d, "\mathrm{d}"] & D^i \arrow[r, "\beta"] \arrow[d, "\mathrm{d}"] & E^i \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\
0 \arrow[r] & C^{i+1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{d}"] & D^{i+1} \arrow[r, "\beta"] \arrow[d, "\mathrm{d}"] & E^{i+1} \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\
0 \arrow[r] & C^{i+2} \arrow[r, "\alpha"] \arrow[d, "\mathrm{d}"] & D^{i+1} \arrow[r, "\beta"] \arrow[d, "\mathrm{d}"] & E^{i+2} \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

such that all squares commute, the columns are the cochain complexes (may have non-trivial cohomology), and the rows are exact, i.e. Im$(\alpha) = \ker(\beta)$*, plus* α *is injective, and* β *is surjective. [The maps into or out of 0 are the obvious ones.]*

Then we have the usual results from Algebraic Topology:

Lemma 2.6 (Snake Lemma). *If* $0 \to C^\bullet \to D^\bullet \to E^\bullet \to 0$ *is a s.e.s of cochain complex, then* $\exists$ *an associated* ***long exact sequence (l.e.s)*** *of cohomology groups:*

$$
\begin{tikzcd}
\cdots \arrow[r] & H^i(C^\bullet) \arrow[r, "\alpha^*"] & H^i(D^\bullet) \arrow[r, "\beta^*"] & H^i(E^\bullet) \arrow[dll, rounded corners, "\delta"] \\
 & H^{i+1}(C^\bullet) \arrow[r, "\alpha^*"] & H^{i+1}(D^\bullet) \arrow[r] & \cdots
\end{tikzcd}
$$

where the connecting map δ *is built algebraically.*

Proof. See Algebraic Topology (Part III).

□

Proposition 2.4 (Mayer-Vietoris). *If* $M = U \cup V$ *is a union of open sets, then* $\exists$ *a s.e.s of cochain complexes:*

$$0 \to \Omega^\star(M) \xrightarrow{\alpha} \Omega^\star(U) \oplus \Omega^\star(V) \xrightarrow{\beta} \Omega^\star(U \cap V) \to 0$$

where $\alpha(\omega) = (\,\omega|_U\,,\,\omega|_V)$ *and* $\beta(\tilde{\alpha}, \tilde{\beta}) = \tilde{\alpha}|_{U\cap V} - \tilde{\beta}|_{U\cap V}$*.*

Proof. We just need to check exactness of the given maps in the above s.e.s: exactness is clear exact perhaps at the last stage, i.e. we need to justify that β is surjective.

Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the (2-set covering) $\{U, V\}$. Then given $\omega \in \Omega^k(U \cap V)$, define:

$$\omega_U = \begin{cases} \rho_V \cdot \omega & \text{on } U \cap V \\ 0 & \text{on } U \backslash (U \cap V), \end{cases} \quad \text{and} \quad \omega_V = \begin{cases} \rho_U \cdot \omega & \text{on } U \cap V \\ 0 & \text{on } V \backslash (U \cap V). \end{cases}$$

Then $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$. Then as $\rho_U + \rho_V = 1$, we have

$$\beta(\omega_U, -\omega_V) = \rho_V \cdot \omega + \rho_U \cdot \omega = \omega$$

i.e. $\omega \in \text{Im}(\beta)$, and so hence β is surjective.

□

Note: Combining Mayer-Vietoris and the Snake Lemma, we see that if $M = U \cap V$ is the union of two open sets, then we get an associated l.e.s on cohomology (given by the Snake Lemma):

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(M) \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(U) \oplus \mathrm{H}^i_{\mathrm{dR}}(V) \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(U \cap V) \arrow[dll, rounded corners] \\ & \mathrm{H}^{i+1}_{\mathrm{dR}}(M) \arrow[r] & \cdots & \end{tikzcd}$$

which is called the **Mayer-Vietoris sequence**. This is useful for calculating the cohomology of more complex objects from simpler ones.

Observe: If $\cdots \to A \xrightarrow{\lambda} B \xrightarrow{\mu} C \to \cdots$ is a piece of a l.e.s of finite dimensional vector spaces (so in particular, $\ker(\mu) = \text{Im}(\lambda)$), then applying the rank-nullity theorem to λ, μ gives:

$$\text{rank}(B) \leq \text{rank}(A) + \text{rank}(C).$$

If M is of rank 1, then $M \cong \mathbb{R}^n$ and we already know that $\mathrm{H}^\star_{\mathrm{dR}}(M)$ is finite dimensional, as de Rham cohomology is invariant under diffeomorphisms, and

$$\mathrm{H}^\star_{\mathrm{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } \star = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If M is of type k, so that $M = U_1 \cup \cdots \cup U_k$, with each $U_i \cong \mathbb{R}^n$, and with the iterated interiors $U_I = U_{i_1} \cap \cdots \cap U_{i_m}$ are diffeomorphic to discs or are empty.

Then we have $M = U \cup V$, where $U := U_1 \cong \mathbb{R}^n$ and $V := U_2 \cup \cdots \cup U_k$ is of type $k-1$ (as covered by $U_2, \ldots, U_k$). Also, $U \cap V$ is of type $k-1$, covered by $U \cap U_2, \ldots, U \cap U_k$.

Hence by induction on the type of M, the Mayer-Vietoris sequence shows that $\mathrm{H}^*_{\mathrm{dR}}(M)$ is finite dimensional when M has finite type, which is a step closer to showing that de Rham cohomology is finite dimensional.

We could develop more theory related to this - but we leave it to the Part III Algebraic Topology course.

2.5. **Moser's Theorem.**

Recall the Lie derivative: if $X \in \Gamma(TM)$ is a vector field, then for $f \in C^\infty(M)$, we define

$$\mathfrak{L}_X(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \varphi_t) = X \cdot f$$

where φ_t is the flow of X. If $Y \in \Gamma(TM)$ is another vector field, then we also have

$$\mathfrak{L}_X(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_t)_*(Y) = [X, Y].$$

Here, as φ_t is a flow, it has a well-defined action:

$$(\varphi_t)_* : \Gamma(TM) \to \Gamma(TM) \quad \text{and} \quad (\varphi_t)^* : \Gamma(T^*M) \to \Gamma(T^*M)$$

and so hence we get a well-defined action

$$\varphi_t : \Gamma(\Lambda^i T^*M) \to \Gamma(\Lambda^i T^*M).$$

So hence we can differentiate **any** (k,l)-tensor, and so in particular differential forms, on a flow, via the Lie derivative $\mathfrak{L}_X$.

Definition 2.10. *If $X \in \Gamma(TM)$ is a vector field and $\omega \in \Omega^r(M)$, then the **contraction of ω** with respect to X, denoted $\iota_X(\omega) \in \Omega^{r-1}(M)$, is defined by:*

$$\iota_X(\omega)(Y_1, \ldots, Y_{r-1}) := \omega(X, Y_1, \ldots, Y_{r-1}),$$

for vector field $Y_1, \ldots, Y_{r-1}$.

If $f \in C^\infty(M) = \Omega^0(M)$, then we define $\iota_X(f) = 0$.

*We call $\iota_X(\omega)$ the **interior product** of ω with X.*

Lemma 2.7. *We have the following properties of $\mathfrak{L}_X$ and ι_X:*

(i) Any Lie derivative is a derivation, i.e.

$$\mathfrak{L}_X(T \otimes S) = \mathfrak{L}_X(T) \otimes S + T \otimes \mathfrak{L}_X(S).$$

(ii) If $\omega, \omega' \in \Omega^\star(M)$ and $X \in \Gamma(TM)$, then we have the following Liebniz property:

$$\iota_X(\omega \wedge \omega') = (\iota_X\omega) \wedge \omega' + (-1)^{|\omega|}\omega \wedge (\iota_X\omega').$$

(iii) ι_X satisfies Poincaré: $\iota_X \circ \iota_X = 0$

(iv) $\mathfrak{L}_X(\mathrm{d}\omega) = \mathrm{d}(\mathfrak{L}_X(\omega))$ if $\omega \in \Omega^\star(M)$, $X \in \Gamma(TM)$, i.e. Lie derivative and exterior derivative commute.

*(v) (**Cartan's Magic Formula**) We have*

$$\mathfrak{L}_X\omega = \iota_X(\mathrm{d}\omega) + \mathrm{d}(\iota_X(\omega))$$

for $X \in \Gamma(TM)$ and $\omega \in \Omega^\star(M)$.

Proof. (i): We know that, since $\varphi_0 = \text{id}$:

$$\mathfrak{L}_X(T\otimes S)|_p = \lim_{t\to 0}\frac{\varphi_t^*(T\otimes S)|_{\varphi_t(p)} - (T\otimes S)|_p}{t}$$

$$= \lim_{t\to 0}\frac{1}{t}\left(\varphi_t^*T|_{\varphi_t(p)}\otimes\varphi_t^*S|_{\varphi_t(p)}\underbrace{-\varphi_t^*T|_{\varphi_t(p)}\otimes S|_p + \varphi_t^*T|_{\varphi_t(p)}\otimes S|_p}_{\text{i.e. add and subtract same thing}} - T|_p\otimes S|_p\right)$$

$$= \lim_{t\to 0}\varphi_t^*T|_{\varphi_t(p)}\otimes\left(\frac{\varphi_t^*S|_{\varphi_t(p)} - S|_p}{t}\right) + \lim_{t\to 0}\left(\frac{\varphi_t^*T|_{\varphi_t(p)} - T|_p}{t}\right)\otimes S|_p$$

$$= T|_p\otimes\mathfrak{L}_X(S)|_p + \mathfrak{L}_X(T)|_p\otimes S|_p$$

by the definition of the Lie derivative, and by continuity of φ_t. So done.

(ii): The result will follow from the following claim:

Claim:

$$\iota_X(\omega_1\wedge\cdots\wedge\omega_k) = \sum_{i=1}^k(-1)^{i-1}\omega_i(X)\cdot\omega_1\wedge\cdots\wedge\underbrace{\hat{\omega}_i}_{\text{i.e. omit}}\wedge\cdots\wedge\omega_k$$

where the ω_i are 1-forms.

Given this claim, using linearity and the Liebnitz rule, we can reduce (ii) to the case of $\omega = f\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_p}$, $\omega' = g\cdot dx_{j_1}\wedge\cdots\wedge x_{j_q}$, and then conclude, as this is simple to prove [**Exercise** to check].

Proof of Claim. Fix $X_1,\dots,X_k$ vector fields with $X = X_1$. Then,

$$\iota_X(\omega_1\wedge\cdots\wedge\omega_k)(X_2,\dots,X_k) = (\omega_1\wedge\cdots\wedge\omega_k)(X_1,X_2,\dots,X_k) = \det((\omega_i(X_j))_{i,j}).$$

Write $A_{ij} = \omega_i(X_j)$ for this $k\times k$ matrix. But then let $A^{(r,s)}$ be the matrix obtained by removing row r and column s from A. Then by expanding the determinant of A along the first column, we have:

$$\det(A) = \sum_{i=1}^k\omega_i(X_1)\cdot(-1)^{i-1}\det(A^{(i,1)})$$

$$= \sum_{i=1}^k(-1)^{i-1}\omega_i(X)\underbrace{(\omega_1\wedge\cdots\wedge\hat{\omega}_i\wedge\cdots\wedge\omega_k)(X_1,\dots,X_k)}_{\text{matrix minor}}.$$

So combining these two calculations proves the claim, and so done. □

(iii): Easy by the antisymmetry of differential forms, i.e. $\omega(X,X,\cdots) = 0$.

(iv): This holds since we know $d(f^*\omega) = f^*(d\omega)$ for $f: M\to N$ a smooth map.

(v): We know that $\mathfrak{L}_X$ and $\iota_X\circ d + d\circ\iota_X$ are derivations which commute with d (by (iv) for $\mathfrak{L}_X$, and direct check for other). Therefore it suffices to prove that they coincide on $\Omega^0(M) = C^\infty(M)$ (as we can reduce to this case by writing in a basis and using the properties above, as we can split up wedge products by (i) and (ii)). So hence, for $f \in C^\infty(M)$ we have:

$$\iota_X(\mathrm{d}f) + \mathrm{d}(\underbrace{\iota_X f}_{=0}) = \iota_X(\mathrm{d}f) := \mathrm{d}f(X) = X \cdot f = \mathfrak{L}_X(f),$$

and so done.

□

Recall: A volume form is a nowhere zero n-form. So given a volume form $\omega \in \Omega^n(M)$, we get a notion of **volume-preserving diffeomorphism**, i.e.

$$\mathrm{Diff}_{\mathrm{vol}}(M) := \{f \in \mathrm{Diff}(M) : f^*\omega = \omega\}.$$

The main result here is the relation of volume forms in the same cohomology class, which is given by Moser's theorem:

Theorem 2.3 (Moser's Theorem). *Let M be a closed manifold and ω_0, $\omega_1 \in \Omega^n(M)$ volume forms on M such that*

$$\int_M \omega_0 = \int_M \omega_1$$

(i.e. by Theorem 2.2, this is just saying that ω_0, ω_1 are in the same de Rham cohomology class in $H^n_{dR}(M)$). Then, $\exists$ a diffeomorphism $\psi : M \to M$ which is smoothly isotopic to the identity such that

$$\psi^*\omega_1 = \omega_0$$

i.e. these volume forms are "the same".

Remark: The proof will show that if $f \in \mathrm{Diff}_{\mathrm{vol}}(M, \omega_0)$, and if f is smoothly isotopic to the identity, then in fact f is isotopic to the identity through volume-preserving diffeomorphisms (i.e. $\mathrm{Diff}_{\mathrm{vol}}(M,\omega) \hookrightarrow \mathrm{Diff}(M)$ is a weak homotopy equivalence).

Recall: Smoothly isotopic means that $\exists$ a path $(f_t)_t$ of diffeomorphisms such that $f_0 = f_1$, $f_1 = \mathrm{id}_M$, with $f_t \in \mathrm{Diff}(M)$ for all t, and $F : M \times [0,1] \to M$, sending $(m,t) \to f_t(m)$, is smooth.

Proof of Moser's Theorem. The linear interpolation $\omega_t = t\omega_0 + (1-t)\omega_1$ is (clearly) a path of volume forms. We seek a family $(\psi_t)_t \subset \mathrm{Diff}(M)$ such that $\psi_t^*\omega_t = \omega_0$, and $\psi_0 = \mathrm{id}_M$.

If we have a path of diffeomorphisms with $\psi_0 = \mathrm{id}$, then there is an associated (time-dependent) vector field $X_t \in \Gamma(TM)$ such that:

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t}(p) = X_t|_{\psi_t(p)}, \quad \text{i.e.} \quad X_t|_p := \frac{\mathrm{d}\psi_t}{\mathrm{d}t}\left(\psi_t^{-1}(p)\right)$$

i.e. we can integrate each ψ_t to get a flow, and patch the flows together for each ψ_t.

But:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega_t) = \psi_t^*\left(\mathfrak{L}_{X_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t}\right)$$

where we have used the chain rule (as $\psi_t^* \omega_t = \omega_t(\mathrm{d}\psi_t))$[(i)]. But then using Cartan's magic formula, noting that $\mathrm{d}(n\text{-form}) = 0$, gives this equals

$$= \psi_t^*(\mathrm{d}(\iota_{X_t}\omega_t) + \mathrm{d}\alpha)$$

where $\mathrm{d}\alpha = \frac{\mathrm{d}\omega_t}{\mathrm{d}t} = \omega_0 - \omega_1$. Now such an α exists since $\int_M (\omega_0 - \omega_1) = 0$, and so hence $[\omega_0 - \omega_1] = 0$ in $\mathrm{H}^n_{\mathrm{dR}}(M)$ (from Theorem 2.2), and so $\omega_0 - \omega_1$ is exact. So hence we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega_t) = \psi_t^*(\mathrm{d}(\iota_{X_t}\omega_t + \alpha)).$$

Exercise: Show that the map $\Gamma(TM) \to \Omega^{n-1}(M)$, sending $X \longmapsto \iota_X(\omega)$, is an isomorphism, if ω is a volume form on a closed manifold.

So hence if $\psi_t^*\omega_t = \omega_0$ is a constant form for all t, then clearly $\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega_t) = 0$.

So hence to achieve this, by the above equation, it suffices to choose vector fields X_t such that

$$\iota_{X_t}\omega_t + \alpha = 0, \tag{2.1}$$

and then to **define** ψ_t by flowing X_t.

But the ω_t are all volume forms, and so hence by the above exercise, we can solve (2.1) for the X_t.

Then on a closed manifold we know every vector field is complete, and so we can **define** ψ_t by the flow of X_t, i.e.

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t} = X_t|_{\psi_t}.$$

Then by construction, we have (from the above calculations), $\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega_t) = 0$, i.e. $\psi_t^*\omega_t =$ constant with t. But then by evaluation at $t = 0$, and $\psi_0 = \mathrm{id}$, we see that this constant is ω_0.

So then take $\psi = \psi_1$. Then this has $\psi^*\omega_1 = \psi_1^*\omega_1 = \omega_0$, and ψ is smoothly isotopic to the identity, via the ψ_t. So done.

$\square$

The above method is called **Moser's method**.

[(i)]This formula comes from the fact that, in general we have for a function $f = f(x,y)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t,t) = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=t} f(x,t) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{y=t} f(t,y).$$

So here we have $f(x,y) = \psi_x^*\omega_y$ and this can be used to show the above formula (using the expressions/definition for the Lie derivative - **Exercise** to check.

3. Connections

We would like to be able to differentiate sections of vector bundles, not just of the tangent bundle (in the case where the vector bundle is the tangent bundle, we have differential forms $\Omega^i(M)$ and the Lie derivative and exterior derivative, i.e. everything as we have studied it so far. We want to generalise this).

Recall that if $E \to M$ is a smooth vector bundle, then we have:

- $\Omega^0(M) = \Gamma(E)$, is the vector space of global sections of E (i.e. smooth E-valued functions on M)
- $\Omega^i(E) = \Gamma(E \otimes \Lambda^i(T^*M))$, is the space of E-valued differential i-forms.

We hence we can think of a E-valued differential i-form at each point $p \in M$ as a map $\underbrace{T_pM \times \cdots \times T_pM}_{i \text{ times}} \to E_p$. This is because at each $p \in M$, the section at p is valued in $(E \otimes \Lambda^i(T^*M))_p = E_p \otimes \Lambda^i(T_p^*M) \cong (\{p\} \times \mathbb{R}^k) \otimes \Lambda^i(T_p^*M)$, where $k = \text{rank}(E)$. So hence this can be written as an element of the form $v \otimes \omega$, for $v \in E_p$ and $\omega \in \Omega^i(M)$ a differential i-form on M (in the usual sense). Hence this is a map on $T_pM \times \cdots \times T_pM$, which, evaluates to $(v \otimes \omega)(x) = v \otimes \underbrace{(\omega(x))}_{\in \mathbb{R}}$, which can be identified by an element of E_p.

Hence this shows that if we take the globally trivial bundle $E = M \times \mathbb{R}$, E-valued differential forms of this type are simply just differential forms of the type considered before (as each fibre is just a copy of $\mathbb{R}$).

Definition 3.1. *A **connection** A on E is a linear operator*
$$\mathrm{d}_A : \Omega^0(E) \to \Omega^1(E)$$
such that it obeys a Leibniz property:
$$\mathrm{d}_A(f \cdot s) = s \otimes \mathrm{d}f + f \cdot \mathrm{d}_A(s)$$
for each $f \in C^\infty(M)$ and $s \in \Omega^0(E) = \Gamma(E)$.

So a connection (or also called a **connexion**) is like a generalisation of the usual exterior derivative to more general vector bundles (instead of just the tangent bundle) (it satisfies the same properties of the exterior derivative, just without the extra properties for higher order differential forms). We therefore hope to define generalisations of a connection to $\Omega^i(E)$, as we did for the exterior derivative.

Let $U \subset M$ be a trivialising open neighbourhood of E. So, $E|_U \cong U \times \mathbb{R}^k$, where $k = \text{rank}(E)$. So let $e_1, \ldots, e_k$ be a local basis of sections of $E|_U$. Then, we know:
$$\mathrm{d}_A(e_i) \in \Omega^1(E|_U) = \Gamma((E \otimes T^*M)|_U) = \Gamma(E|_U \otimes T^*U)$$
from properties of the tensor product. So hence using this basis of $E|_U$, we know that we can write:
$$\mathrm{d}_A(e_i) = \sum_{j=1}^k e_j \otimes \theta_{ji}$$

where $\theta_{ji} \in \Omega^1(U)$ is a 1-form on U (these are just arbitrary 1-forms, not in terms of a basis of $\Omega^1(U)$). This holds locally, in U. We say that $(\theta_{ij})_{ij}$ is the **connexion matrix** for the connexion A in the open set U - since d_A is linear, it is determined by them. So since any local section of $E|_U$ is of the form: $s = \sum_{i=1}^k s_i e_i$, where $s_i \in C^\infty(U)$, the Leibniz property of d_A gives:

$$\mathrm{d}_A(s) = \sum_i \mathrm{d}_A(s_i e_i) = \sum_i (e_i \otimes \mathrm{d}s_i + s_i \mathrm{d}_A e_i) = \sum_i (\mathrm{d}s_i + \sum_j s_j \theta_{ji}) \otimes e_i$$

and so we see that locally, d_A "acts" as "$\mathrm{d}_A = \mathrm{d} + \theta$".

We now wish to know how θ changes under coordinates changes, so we know how connexions change. So if $e'_1, \ldots, e'_k$ is another local basis of sections, then $\exists$ a map $\psi : U \to \mathrm{GL}_k(\mathbb{R})$ such that:

$$e'_j|_p = \sum_{i=1}^k \psi_{ji} e_i|_p$$

for $p \in U$. So the ψ_{ji} gives a basis change, and the following lemma tells us how the connexion matrix changes:

Lemma 3.1. *In the $\{e'_i\}$ basis, the connexion matrix is:*

$$\theta' = (\mathrm{d}\psi) \cdot \psi^{-1} + \psi \theta \psi^{-1}.$$

Equivalently, given an open trivialising cover of M for E, to define a connexion A on E, it suffices to give matrices $(\theta_{ij})_{ij}$ of 1-forms on open sets which satisfy the above compatibility condition/transformation law.

Proof. Let $\mathfrak{F} = (e_1, \ldots, e_k)$ be our original frame ($\equiv$ local trivialising basis). Then in this basis, we know:

$$\mathrm{d}_A(\mathfrak{F}) = \theta(\mathfrak{F}) \cdot \mathfrak{F},$$

i.e. this is the usual expression "$X \cdot f$", except know we have a matrix of 1-forms X and this is matrix multiplication.

Now consider the new frame, which by the above can be written: $\mathfrak{F}' = \psi \cdot \mathfrak{F}$ (in the same way as above). Then, by the Leibniz property,

$$\begin{aligned}
\mathrm{d}_A(\mathfrak{F}') &= \mathrm{d}_A(\psi \cdot \mathfrak{F}) \\
&= \mathrm{d}\psi \cdot \mathfrak{F} + \psi \cdot \mathrm{d}_A(\mathfrak{F}) = \mathrm{d}\psi \cdot \mathfrak{F} + \psi \cdot (\theta(\mathfrak{F}) \cdot \mathfrak{F}) \\
&= [\mathrm{d}\psi \cdot \psi^{-1} + \psi \theta(\mathfrak{F}) \psi^{-1}](\psi \cdot \mathfrak{F}) \\
&= \theta'(\mathfrak{F}'),
\end{aligned}$$

where $\theta' = \mathrm{d}\psi \cdot \psi^{-1} + \psi \theta \psi^{-1}$. So this is the connexion matrix for the frame $\mathfrak{F}'$.

□

Remark: Essentially everything with connexions revolves around working locally on a trivialising cover, where we know $\mathrm{d}_A = \mathrm{d} + \theta$.

Now some standard useful properties about the space of connexions on E:

Lemma 3.2. *We have*

(i) Every such E admits some connexion

(ii) The space of connexions on E, denoted $\mathbb{A}_E$, is an affine space for the vector space $\Omega^1(End(E))$ (which is defined as above).

Proof. (i): Take a trivialising open cover $\{U_\alpha\}_\alpha$ and a subordinate partition of unity $\{\lambda_\alpha\}_\alpha$.

Then on each U_α, choose a matrix of 1-forms θ_α and define a connexion on $E|_{U_\alpha}$ by θ_α, i.e. set $\mathrm{d}_{A_\alpha} = \mathrm{d} + \theta_\alpha$ with respect to our chosen trivialisation.

Note that the defining conditions for being a connexion are preserved by taking convex combinations. In particular, this shows that

$$\mathrm{d}_A = \sum_\alpha \lambda_\alpha \mathrm{d}_{A_\alpha}$$

defines some connexion on all of E.

(ii): If $A, B \in \mathbb{A}_E$, then we know:

$$\mathrm{d}_A(f \cdot s) = \mathrm{d}f \otimes s + f\,\mathrm{d}_A(s)$$

$$\mathrm{d}_B(f \cdot s) = \mathrm{d}f \otimes s + f\,\mathrm{d}_B(s).$$

So we see that,

$$(\mathrm{d}_A - \mathrm{d}_B)(f \cdot s) = f \cdot (\mathrm{d}_A - \mathrm{d}_B)(s),$$

i.e. the difference between connexions, $\mathrm{D} := \mathrm{d}_A - \mathrm{d}_B$, is a $C^\infty(M)$-module map (as $\mathrm{D}(f \cdot s) = f \cdot \mathrm{D}(s)$ by the above [See Example Sheet 2, Q9, for a comparison].

So hence $\mathrm{d}_A - \mathrm{d}_B$ comes from a bundle map in $\Gamma(\mathrm{Hom}(E, E \otimes T^*M))$. But then note, from properties of Hom and $\otimes$ (see Q1 on Example Sheet 2) we see:

$$\Gamma(\mathrm{Hom}(E, E \otimes T^*M)) = \Gamma(E^* \otimes E \otimes T^*M) = \Gamma(\mathrm{Hom}(E,E) \otimes T^*M) =: \Omega^1(\mathrm{End}(E)),$$

and $\mathrm{Hom}(E,E) = \mathrm{End}(E)$. So hence the difference between connexions is in $\Omega^1(\mathrm{End}(E))$, and hence $\mathbb{A}_E$ is an affine (sub)space for $\Omega^1(\mathrm{End}(E))$ (i.e. think of a plane in $\mathbb{R}^2$ which doesn't pass through the origin: it itself is not a linear subspace as it does not contain 0, but it is an affine subspace).

□

Now we can prove that, just like the exterior derivative, we get extensions of connexions to higher differential forms:

Lemma 3.3. *If A is a connexion on $E \to M$, then $\exists$ a natural linear operator*

$$\mathrm{d}_A : \Omega^i(E) \to \Omega^{i+1}(E)$$

for each i.

Proof. Suppose as always, we have a trivialising cover $\{U_\alpha\}_{\alpha\in I}$ for E. So a section of E is locally given by matrix-valued functions ω_α over U_α, which transform as: $\omega_\alpha = \psi_{\alpha\beta}\omega_\beta$, where $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$. So locally, we can define

$$(\mathrm{d}_A\omega)_\alpha := \mathrm{d}\omega_\alpha + \theta_\alpha \wedge \omega_\alpha.$$

This expressions makes sense provided it transforms well (i.e. since we have only defined this on each subset of a trivialising cover, we need to ensure it is compatible on overlaps). So note, changing coordinates $\beta \mapsto \alpha$ by acting by $\psi_{\alpha\beta}$ and vice versa, we have:

$$\begin{aligned}(\mathrm{d}_A\omega)_\beta &= \psi_{\beta\alpha}(\mathrm{d}_A\omega)_\alpha = \psi_{\beta\alpha}(\mathrm{d}\omega_\alpha + \theta_\alpha \wedge \omega_\alpha)\\ &= \psi_{\beta\alpha}(\mathrm{d}(\psi_{\alpha\beta}\omega_\beta)) + \psi_{\beta\alpha}\theta_\alpha \wedge (\psi_{\alpha\beta}\omega_\beta)\\ &= \underbrace{\psi_{\beta\alpha}\psi_{\alpha\beta}}_{=\mathrm{id}}\mathrm{d}\omega_\beta + \psi_{\beta\alpha}(\mathrm{d}\psi_{\alpha\beta})\cdot\omega_\beta + (\psi_{\beta\alpha}\theta_\alpha\psi_{\alpha\beta}) \wedge \omega_\beta\\ &= \mathrm{d}\omega_\beta + \underbrace{\left((\mathrm{d}\psi_{\alpha\beta})\psi_{\alpha\beta}^{-1} + \psi_{\alpha\beta}\theta_\alpha\psi_{\alpha\beta}^{-1}\right)}_{=\theta_\beta \text{ from Lemma 3.1}}\cdot\omega_\beta = \mathrm{d}\omega_\beta + \theta_\beta \cdot \omega_\beta.\end{aligned}$$

Hence the expression for $(\mathrm{d}_A\omega)_\alpha$ makes good sense, even if ω_α are local E-valued forms of every degree. This shows that d_A is well-defined on $\Omega^i(E)$, by using the above definition.

□

Note: Unlike for the exterior derivative, the composition

$$\Omega^i(E) \xrightarrow{\mathrm{d}_A} \Omega^{i+1}(E) \xrightarrow{\mathrm{d}_A} \Omega^{i+2}(E)$$

need not vanish.

So the above shows us that we can **always differentiate differential forms via a connexion**.

Definition 3.2. *The map $\Omega^i(E) \to \Omega^{i+2}(E)$, defined by $\alpha \longmapsto \mathrm{d}_A(\mathrm{d}_A(\alpha))$ has the form:*

$$\alpha \longmapsto F_A \wedge \alpha$$

*for some $F_A \in \Omega^2(End(E))$ (defined as before), which is called the **curvature** (of the connexion) A.*

So this definition tells us that here, $\mathrm{d}_A^2 = F_A\wedge$, where F_A is the curvature.

Lemma 3.4. *Locally on a trivialising cover, we have*

$$(F_A)_\alpha = \mathrm{d}\theta_\alpha + \theta_\alpha \wedge \theta_\alpha$$

i.e. we can directly calculate the curvature of a connexion from the connexion matrix. [This also shows us that F_A is C^∞-linear in both entries.]

Proof. Note that locally we know that $\mathrm{d}_A = \mathrm{d} + \theta_\alpha \wedge$, and so:

$$\begin{aligned}
(\mathrm{d}_A^2(s))_\alpha &= \mathrm{d}_A(\mathrm{d}s_\alpha + \theta_\alpha \wedge s_\alpha)\\
&= \mathrm{d}(\mathrm{d}s_\alpha + \theta_\alpha \wedge s_\alpha) + \theta_\alpha \wedge (\mathrm{d}s_\alpha + \theta_\alpha \wedge s_\alpha)\\
&= \underbrace{\mathrm{d}^2 s_\alpha}_{=0 \text{ as } \mathrm{d}^2=0} + \mathrm{d}(\theta_\alpha \wedge s_\alpha) + \theta_\alpha \wedge \mathrm{d}s_\alpha + \theta_\alpha \wedge (\theta_\alpha \wedge s_\alpha)\\
&= \mathrm{d}\theta_\alpha \wedge s_\alpha - \theta_\alpha \wedge \mathrm{d}s_\alpha + \theta_\alpha \wedge \mathrm{d}s_\alpha + (\theta_\alpha \wedge \theta_\alpha) \wedge s_\alpha\\
&= (\mathrm{d}\theta_\alpha + \theta_\alpha \wedge \theta_\alpha) \wedge s_\alpha,
\end{aligned}$$

where we have used the Leibniz rule for the exterior derivative, d. So indeed, we see that d_A^2 takes this form, and $(F_A)_\alpha = \mathrm{d}\theta_\alpha + \theta_\alpha \wedge \theta_\alpha$.

Now, we need to check that F_A obeys the correct transformation law for $\mathrm{End}(E)$ (i.e. agrees on overlaps of the trivialising cover). So note,

$$\begin{aligned}
(F_A)_\beta &= \mathrm{d}\theta_\beta + \theta_\beta \wedge \theta_\beta\\
&= \mathrm{d}(\psi_{\beta\alpha}\mathrm{d}\psi_{\alpha\beta} + \psi_{\beta\alpha}\theta_\alpha\psi_{\alpha\beta}) + \underbrace{(\psi_{\beta\alpha}\mathrm{d}\psi_{\alpha\beta} + \psi_{\beta\alpha}\theta_\alpha\psi_{\alpha\beta})}_{=\theta_\beta} \wedge (\psi_{\beta\alpha}\mathrm{d}\psi_{\alpha\beta} + \psi_{\beta\alpha}\theta_\alpha\psi_{\alpha\beta})
\end{aligned}$$

where we have used the transformation law of θ_α, noting that $\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}$. Then one can check that this equals [**Exercise** to check]

$$= \psi_{\beta\alpha}(\mathrm{d}\theta_\alpha + \theta_\alpha \wedge \theta_\alpha)\psi_{\alpha\beta} = \psi_{\alpha\beta}^{-1}(F_A)_\alpha\psi_{\alpha\beta}.$$

So hence this obeys the correct transformation law, and so done.

□

Remark: A connexion A on E **induces** connexions on associated bundles, i.e. A^* on E^*, the dual bundle. For example,

$$\mathrm{d}_{A^*} : \Omega^0(E^*) \to \Omega^1(E^*) := \Gamma(E^* \otimes T^*M) \quad \text{is given by:}$$
$$(\mathrm{d}_{A^*}(\xi))(s) := \mathrm{d}(\xi(s)) - \xi(\mathrm{d}_A(s)) \quad \text{for } s \in \Gamma(E).$$

Note that for each $\xi \in \Omega^0(E^*) = \Gamma(E^*)$, we know that $\mathrm{d}_{A^*}(\xi)$ should be valued in $\Omega^1(E^*)$, i.e. should be an E^*-valued function on TM. Hence we need to be able to define its action on E. So hence it suffices to define it on sections of E.

Alternatively, if A is given by a connexion matrix θ_α, then A^* has connexion matrix $-\theta_\alpha^T$, since if $\{e_j\}$, $\{e_j^*\}$ are local dual bases for E and E^*, with ψ being the connexion matrix for the $\{e_j^*\}$, then we have $e_j^*(e_i) = \delta_{ij}$ and so:

$$\begin{aligned}
0 = \mathrm{d}(e_j^*(e_i)) &= \left(\mathrm{d}_{A^*}(e_j^*)\right)(e_i) + e_j^*(\mathrm{d}_A(e_i))\\
&= \left(\sum_k e_k^* \otimes \psi_{kj}\right)(e_i) + e_j^*\left(\sum_k e_k \otimes \theta_{ki}\right)\\
&= \sum_k e_k^*(e_i)\psi_{kj} + \sum_k e_j^*(e_k) \otimes \theta_{ki}\\
&= \sum_k \delta_{ki}\psi_{kj} + \sum_k \delta_{jk}\theta_{ki}\\
&= \psi_{ij} + \theta_{ji}
\end{aligned}$$

i.e. $\psi_{ij} = -\theta_{ji}$.

Remark: Alternatively, using upper and lower indices, this can be written as:

$$\psi^i_j = -\theta^i_j$$

where $d_A e^i = \psi^i_j e^j$ and $d_A e_i = \theta^j_i e_j$. This is handy, since the actual indices don't/can't change and so gives a much easier way of remembering it. [Even though I don't use upper and lower indices in these notes, they are much easier to work with!]

Similarly, $\exists$ induced connexions on direct sums and tensor products, given by:

- On direct sums, $E \oplus F$: $d_{A\oplus B}(s,t) = (d_A s, d_B t)$
- On tensor products, $E \otimes F$: $d_{A\otimes B}(s \otimes t) = d_A(s) \otimes t + s \otimes d_B(t)$.

Both of these can be checked to be connexions on the relevant spaces [**Exercise** to check].

Lemma 3.5 (The 2nd Bianchi Identity). *If A is a connexion on E, then:*

$$d_{A^*\otimes A}(F_A) = 0$$

for $F_A \in \Omega^2(End(E))$ the curvature of A.

Informally, as "$F_A = d_A^2$", this says that: "$d_A^3 = 0$" is always true (instead of $d^2 = 0$ for the exterior derivative).

Proof. If $\psi \in \Omega^s(\mathrm{Hom}(E,F))$ and the bundles E,F have connexions A,B respectively, with associated connexion matrices $\theta_\alpha, \Theta_\alpha$ respectively, then by the definition of $d_{A^*\otimes B}$ (using the above formulae - **Exercise** to check) in $\mathrm{Hom}(E,F)$, we have:

$$d_{A^*\otimes B}(\varphi)_\alpha = d\varphi_\alpha + \theta_\alpha \wedge \varphi_\alpha + (-1)^{|\varphi|}\varphi_\alpha \wedge \theta_\alpha.$$

For us, we know $(F_A)_\alpha = d\theta_a + \theta_a \wedge \theta_\alpha$, where θ_α is the connexion matrix for A in E. So hence taking $A = B$ and $\varphi = F_A$ in the above gives:

$$\begin{aligned} d_{A^*\otimes A}(F_A)_\alpha &= d(\theta_\alpha \wedge \theta_\alpha + d\theta_\alpha) + \theta_\alpha \wedge (d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha) - (d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha) \wedge \theta_\alpha \\ &= d(\theta_\alpha \wedge \theta_\alpha) + \theta_\alpha \wedge d\theta_\alpha - d(\theta_\alpha) \wedge \theta_\alpha = 0 \end{aligned}$$

where we have used the Leibniz property of d and the fact that $d^2 = 0$ for the usual exterior derivative. So done.

$\square$

Definition 3.3. *We call the map d_A **covariant differentiation** with respect to the connexion A.*

*If $\varphi \in \Omega^0(E) = \Gamma(E)$ satisfies $d_A\varphi = 0$, then we say that φ is **covariant constant** or **parallel**.*

Now we prove a result about when $\exists$ a local basis of covariant constant sections of E.

Theorem 3.1. *Let A be a connexion on $E \to M$. Then:*

$$F_A \equiv 0, \text{ i.e. } \mathrm{d}_A^2 = 0 \iff \forall m \in M,\ \exists \text{ a neighbourhood } U \ni m \text{ over which } E \text{ has a basis of covariant constant sections,}$$

i.e. the curvature of the connexion is the only thing stopping us from choosing such a basis of sections.

For the proof of this, we will need the following lemma:

Lemma 3.6. *Let M^n be a manifold, and take $\{\theta_i : 1 \leq i \leq m\}$ be 1-forms, i.e. $\theta_i \in \Gamma(T^*M)$ which are linearly independent. Then, let $V = \{x \in TM : \theta_i(x) = 0\ \forall\ 1 \leq i \leq m\}$ be the associated distribution (i.e. subbundle of TM, zero locus of these forms). Then,*

$$V \text{ is involutive} \iff \exists \text{ 1-forms } \alpha_{ji} \text{ such that } \mathrm{d}\theta_i = \sum_{j=1}^{m} \theta_j \wedge \alpha_{ji} \text{ for each } 1 \leq i \leq m.$$

[Recall that being ***involutive*** *means satisfying the conditions of the F.I.T, Theorem 1.3.]*

Proof of Lemma. Let $X, Y \in V \subset \Gamma(TM)$. Then we know that

$$\theta_i([X,Y]) = \underbrace{X \cdot \theta_i(Y) - Y \cdot \theta_i(X)}_{=0 \text{ as } X,Y \in V} - \mathrm{d}\theta_i(X,Y) = -\mathrm{d}\theta_i(X,Y). \tag{3.1}$$

Then extend $\{\theta_1, \ldots, \theta_m\}$ to a local basis, $\theta_1, \ldots, \theta_m, \ldots, \theta_n$. Then necessarily for some a^i_{jk}, as then the set $\{\theta_j \wedge \theta_k\}_{j,k}$ forms a basis of $\Omega^2(M)$, we must have

$$\mathrm{d}\theta_i = \sum_{j,k=1}^{n} a^i_{jk} \theta_j \wedge \theta_k.$$

Let $\vartheta_1, \ldots, \vartheta_n$ be the dual basis to $\{\theta_i\}_i$ (hence ϑ_i are in TM). So locally, as $\theta_i(\vartheta_j) = \delta_{ij}$, we have that $\vartheta_{m+1}, \ldots, \vartheta_n$ span V (as cannot contain any of the ϑ_i, $i \leq m$, as then one of the θ_i would evaluate to something non-zero). Hence,

$$\begin{aligned}
V \text{ is involutive} &\iff [\vartheta_i, \vartheta_j] \in V \text{ for all } i,j > m \\
&\iff \mathrm{d}\theta_k(\vartheta_i, \vartheta_j) = 0\ \forall k \leq m \text{ when } i,j > m \text{ (by Eq. (3.1) above, by definition of } V) \\
&\iff \sum_{p,q=1}^{n} a^k_{pq} \theta_p \wedge \theta_q(\vartheta_i, \vartheta_j) = 0\ \forall k \leq m,\ \forall i,j > m \\
&\iff \sum_{p,q} a^k_{pq}(\delta_{pi}\delta_{qj} - \delta_{pj}\delta_{qi}) = 0 \text{ (as this is a dual basis)} \\
&\iff a^k_{ij} = a^k_{ji}.
\end{aligned}$$

But then noting that the $\theta_i \wedge \theta_j$ are skew-commutative, and so from the expression from $\mathrm{d}\theta_k$ above, we can wlog assume that $a^k_{ij} = -a^k_{ji}$, and so:

$$V \text{ is involutive} \iff a^k_{ij} = 0\ \forall i,j > m,\ k \leq m$$

$$\iff \mathrm{d}\theta_k = \sum_{i,j=1}^{m} a_{ij}^k \theta_i \wedge \theta_j \text{ for } k \leq m.$$

Hence this proves the theorem by taking $\alpha_{ji} = -\sum_{i=1}^m a_{ij}^k \theta_i$.

□

Proof of Theorem 3.1.

<u>(⇐):</u> If $\exists$ a basis $s_1, \ldots, s_k$ of sections of a rank k bundle E such that $\mathrm{d}_A s_i = 0$, then in this basis, the connexion matrix is $\theta = 0$, and so hence from our local expression of F_A, $(F_A)_\alpha = \mathrm{d}\theta + \theta \wedge \theta = 0$.

<u>(⇒):</u> Let us take local coordinates $x_1, \ldots, x_n$ on M. Take a local basis of sections $s_1, \ldots, s_k$ of E. Then, as before (by definition of the connexion matrix), $\mathrm{d}_A(s_j) = \sum_i s_i \otimes \theta_{ij}$, for local matrix-valued 1-forms.

Extend $x_1, \ldots, x_n$ to a system of local coordinates $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k$ on the total space of E. Then let:

$$\psi_i = \mathrm{d}\lambda_i + \sum_{j=1}^{k} \theta_{ij}\lambda_j,$$

which is a locally-defined 1-form on E. Now,

$$\begin{aligned}
\mathrm{d}\psi_i &= \sum_{j=1}^{k} \big((\mathrm{d}\theta_{ij})\lambda_j - \theta_{ij} \wedge \mathrm{d}\lambda_j\big) \\
&= \sum_{j=1}^{k} \left(\mathrm{d}\theta_{ij} \cdot \lambda_j - \theta_{ij} \wedge \left(\psi_j - \sum_{r=1}^{k} \theta_{jr}\lambda_r \right) \right) \\
&= \underbrace{\sum_{j,r} (\mathrm{d}\theta_{ij}\lambda_j + \theta_{ij} \wedge \theta_{jr}\lambda_r)}_{=(F_A\lambda)_i \text{ by definition of } F_A} + \sum_{j=1}^{k} \psi_j \wedge \theta_{ij}
\end{aligned}$$

where in the second equality we used the expression for ψ_j. So hence as by assumption, $F_A = 0$, we have

$$\mathrm{d}\psi_i = \sum_{j=1}^{k} \psi_j \wedge \theta_{ij}.$$

So by the previous lemma, this shows that if $V = \{x \in TE : \psi_i(x) = 0 \ \forall 1 \leq i \leq k\}$, then V is involutive. So by the Frobenius Integrability Theorem (Theorem 1.3), we get that $\exists$ coordinates $y_1, \ldots, y_{n+k}$ locally on E such that

$$V = \operatorname{span}\left\langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\rangle.$$

Then the **annihilator** of V is clearly just:

$$V^0 := \operatorname{span}\langle \psi_1, \ldots, \psi_k \rangle = \operatorname{span}\langle \mathrm{d}y_{n+1}, \ldots, \mathrm{d}y_{n+k} \rangle.$$

So hence we can write the ψ_i in terms of these y_j, $j \geq n+1$. So write:

$$\psi_i = \mathrm{d}\lambda_i + \sum_{j=1}^{k} \theta_{ij}\lambda_j = \sum_{r=n+1}^{n+k} c_{ir}\mathrm{d}y_r,$$

for some local smooth functions $\{c_{ir}\}_{ir}$. Now fix constants $y_{n+1} = a_1, \ldots, y_{n+k} = a_k$. This then defines $W \subset E$, a local integral submanifold for V (i.e. the space where these coordinates take these constant values). So let $\pi : E \to M$ be the bundle projection. Then:

Claim: $\pi|_W : W \to M$ is a local diffeomorphism.

Proof of Claim. Locally, $\pi(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k) = (x_1, \ldots, x_n)$. So if $v \in \ker(\mathrm{d}\pi)$, then we may write $v = \sum_{i=1}^{k} \alpha_i \frac{\partial}{\partial \lambda_i}$ for some α_i. But if $v \in TW \subset V$, then $\psi_i(v) = 0$ for all i, and so necessarily $\alpha_i = 0$ for all i (by the "shape of" $\mathrm{d}\lambda_i$). So hence $\ker(\mathrm{d}\pi)$ is transverse to TW (as the above shows $\ker(\mathrm{d}\pi) \cap TW = \{0\}$), i.e. there have non-interesting tangent spaces, and so indeed by the implicit function theorem, $\pi|_W$ is a local diffeomorphism.

□

So at $m \in M$, there is a local inverse to $\pi|_W$, which gives functions $\gamma_1(x_1, \ldots, x_n), \ldots, \gamma_k(x_1, \ldots, x_n)$ such that $\gamma(x) \in W$. So hence as y_r is constant here, for $r \geq n+1$, we see $\mathrm{d}y_r = 0$ for all such r, and so hence from our previous expression for ψ_i:

$$\psi_i|_{\gamma_i(x)} = \left(\mathrm{d}\lambda_i + \sum_j \theta_{ij}\lambda_j\right)\Bigg|_{\gamma_i(x)} = \left(\sum_{r=n+1}^{n+k} c_{ir}\mathrm{d}y_r\right)\Bigg|_{\gamma_i(x)} = 0.$$

So in the coordinates $\lambda)i \circ \gamma_i$, $1 \leq i \leq k$, then we see

$$\mathrm{d}(\lambda_i \circ \gamma) + \sum_j \theta_{ij}(\lambda_j \circ \gamma) = 0$$

in a neighbourhood of chosen $m \in M$ at the centre of the local coordinates. So let $f_i = \lambda_i \circ \gamma_i$. Let $s = \sum_{i=1}^{k} f_i s_i$. Then,

$$\mathrm{d}_A(s) = \sum_i \left(\mathrm{d}f_i \otimes s_i + f_i \sum_{j=1}^{k} \theta_{ij} \otimes s_j\right) = 0,$$

and so s is a covariant constant section.

Now by varying the coefficients a (and choice of integral submanifold W), we get a basis of such sections. So done.

□

Remark: (**Holonomy**) Let $E \to M$ be a bundle with connexion A. Suppose $\gamma : [0,1] \to M$ is a smooth path in M. Then we get a pullback connection, $\gamma^* A$, in the pullback bundle $\gamma^* E \to [0,1]$, over $[0,1]$. But then since $\Lambda^2([0,1]) = \{0\}$, we must have $F_{\gamma^* A} = 0$, as the curvature lies in $\Lambda^2(\mathrm{End}([0,1]))$. So by the above theorem, $\forall t \in [0,1]$, we get that $\exists$ a local basis of covariant constant sections.

So if $\text{rank}(E) = k$, and $\{t_1, \ldots, t_k\}$, $\{s_1, \ldots, s_k\}$ are 2 such local bases', then we know we can write:

$$t_i = \sum_{j=1}^{k} c_{ji} s_j$$

for some c_{ji}, for all i. So hence as these are covariant constant, by the Leibniz property of connexions, we know

$$0 = \mathrm{d}_{\gamma^* A}(t_i) = \sum_j \mathrm{d}c_{ji} \cdot s_j + \sum_j c_{ji} \underbrace{\mathrm{d}_{\gamma^* A}(s_j)}_{=0}$$

i.e. $\sum_j \mathrm{d}c_{ji} \cdot s_j = 0$. But then the linear dependence of the s_i implies that $\mathrm{d}c_{ji} = 0$ for all i, j and so hence the c_{ji} are constants.

So, by covering $[0,1]$ by a finite set of open intervals on which we have local bases of covariant constant sections (which we can do by the compactness of $[0,1]$, as can do locally about each point), and then by adjusting the sections on overlaps (which we can do by the above, as the change of basis coefficients are constants), we get that $\exists\ \sigma_1, \ldots, \sigma_k$, a family of **global** covariant constant sections.

So hence if $v = v_0 = \sum_{i=1}^{k} c_i \sigma_i(0) \in E_{\gamma(0)}$ (as the $\sigma_i(0)$ form a basis here), we can consider the extension of v, denoted v_t, to $E_{\gamma(t)}$, for all t, and so in particular we can consider $v_1 = \sum_{i=1}^{k} c_i \sigma_i(1) \in E_{\gamma(t)}$, via

$$v_t = \sum_{i=1}^{k} c_i \sigma_i(t) \in E_{\gamma(t)}.$$

So hence we can push vectors in the fibres of E along paths.

Definition 3.4. *We call v_1 the **parallel transport** of $v = v_0$ along γ (in E).*

So hence parallel transport defines an invertible map $E_{\gamma(0)} \to E_{\gamma(1)}$ (it is invertible since if we consider the reverse curve, $\tilde{\gamma})(t) = \gamma(1-t)$ from $E_{\gamma(1)}$ to $E_{\gamma(0)}$, parallel transport along this gives the inverse.)

In particular, if $\gamma : S^1 \to M$ is a smooth **loop** in M, then parallel transport then gives an automorphism of $E_{\gamma(0)}$ (as $\gamma(0) = \gamma(1)$), which is called the **holonomy of the connexion** A along γ.

So hence by varying the loops γ, we get a map: $C^\infty(S^1, M) =: \underbrace{L_{C^\infty}(M)}_{\text{smooth loops in } M} \to \text{GL}(E_{\gamma(0)})$, whose image is called the **holonomy group**.

If M is connected, then the corresponding conjugacy class of subgroups of $\text{GL}_k(\mathbb{R})$ is well-defined.

[See Example Sheet 2 for the following exercise: "If $F_A \equiv 0$ in M, then in fact the holonomy gives a map: $\pi_1(M, m) \to \text{GL}(E_m)$ for each $m \in M$."]

3.1. **Chern-Weil Theory.**

Recall that we showed if M is a closed (i.e. compact without boundary) oriented n-manifold, then $\text{H}^n_{\text{dR}}(M) \cong \mathbb{R}$. In particular, if $\alpha \in \text{H}^k_{\text{dR}}(M)$, say $\alpha = [\omega]$ for some $\omega \in \Omega^k(M)$, then we know $\mathrm{d}\omega = 0$.

Then if $i : Y \hookrightarrow M$ is a closed oriented k-dimensional submanifold of M, then we can consider the pullback of α to Y under the inclusion i, and consider: $\int_Y \alpha|_Y \in \mathbb{R}$.

This one way to think about a degree k class in $\mathrm{H}^\star_{\mathrm{dR}}(M)$ is as a way of associating real numbers to (closed oriented) k-dimensional submanifolds of M, via the map:

$$\alpha : \{\text{Such } k\text{-dimensional submanifolds}\} \to \mathbb{R}, \quad \text{via} \quad Y \longmapsto \int_Y \alpha|_Y.$$

In another direction, we could view $\mathrm{H}^\star_{\mathrm{dR}}(M)$ as a recipient for invariants of smooth vector bundles over M. We explore this more now. Indeed, let $E \to M$ be a vector bundle, and pick a connexion A in E. So we get the curvature of A, $F_A \in \Omega^2(\mathrm{End}(E)) = \Gamma(\mathrm{End}(E) \otimes \Lambda^2(T^*M))$.

Combining the wedge product $\wedge$ on differential forms with the composition of endomorphisms, we then get a natural notion of $F_A^m \in \Omega^{2m}(\mathrm{End}(E))$: indeed, if we have $\varphi \otimes s \in \Omega^i(\mathrm{End}(E))$, with $\varphi \in \Gamma(\mathrm{End}(E))$, $s \in \Gamma(\Lambda^i(T^*M))$ (which we can write F_A as locally, in the $i = 2$ case), then we can define:

$$(\varphi \otimes s)^m := (\varphi \circ \cdots \circ \varphi) \cdot s \wedge \cdots \wedge s \quad \in \Omega^{mi}(\mathrm{End}(E)).$$

Note that we can then that, viewing this as an endomorphism of M combined with a differential form, if we take the trace of the endomorphism part (in the usual) way, then we are just left with a differential form, i.e. $\mathrm{tr}(\varphi \otimes s)^m \in \Omega^{mi}(M)$ is a i-form on M. So in particular,

$$\mathrm{tr}(F_A^m) \in \Omega^{2m}(M)$$

is a differential form of M. So we can ask all the natural questions about this differential form: is it closed? Is it exact? We find:

Lemma 3.7. *$\forall m \geq 1$, $tr(F_A^m)$ is a closed differential form (under the usual exterior derivative), and so hence it defines an equivalence class $[tr(F_A^m)] \in H^{2m}_{dR}(M)$.*

Proof. In a bit. □

Lemma 3.8. *The equivalence class $[tr(F_A^m)]$ depends on E, but* **not** *on the choice of connexion A in E.*

Proof. In a bit. □

So hence this shows that we get invariants for smooth vector bundles, which lie in the de Rham cohomologies.

Remark: The association map $E \longmapsto [\mathrm{tr}(F_A^m)] =: \mathrm{ch}_M(E) \in \mathrm{H}^{2m}_{\mathrm{dR}}(M)$ is a **characteristic class** for smooth vector bundles, meaning that it is natural in the following sense:

> "If $f : M \to N$ is smooth and $E \to N$, then we get a pullback bundle, $f^*E \to M$ over M, and $\mathrm{ch}_M(f^*E) = f^*(\mathrm{ch}_M(E))$ in $\mathrm{H}^{2m}_{\mathrm{dR}}(M)$"

i.e. we have in this case a well-defined map $f^*: \mathrm{H}^{2m}_{\mathrm{dR}}(N) \to \mathrm{H}^{2m}_{\mathrm{dR}}(M)$ (to see this, we use the pullback connexion in f^*E).

Also, if $E \cong M \times \mathbb{R}^k$ is a trivial bundle, there is a trivial connextion $\mathrm{d}_A \equiv \mathrm{d}$ on E (i.e. $\theta_\alpha \equiv 0$, so get usual exterior derivative), and then $F_A \equiv 0$ (as $\mathrm{d}^2 = 0$), and so we have $\mathrm{ch}_M(E) = 0$, for all $m \geq 1$. So hence this shows:

$$E \cong \text{ a trivial bundle } \Rightarrow \mathrm{ch}_M(E) = 0 \ \forall m \geq 1,$$

or alternatively:

$$\text{If } \mathrm{ch}_M(E) \neq 0 \text{ for some } m \geq 1 \Rightarrow E \ncong \text{ trivial bundle.}$$

Remark: Via taking $E = TM$, one sees that there are **canonical elements** in $\mathrm{H}^\star_{\mathrm{dR}}(M)$ for a smooth manifold M, corresponding to these traces (i.e. the second lemma tell us that this only depends on $E = TM$, which only depends on M). There therefore yields constraints on how $\mathrm{Diff}(M)$ acts on $\mathrm{H}^\star_{\mathrm{dR}}(M)$, and is important, e.g. in the classification of manifolds.

Proof of Lemma 3.7. We give two proofs:

Short Proof: From the definition of the induced connexion on $A^* \otimes A$ and the fact that the trace acts only on the $\mathrm{End}(E)$ part of elements of $\Omega^*(\mathrm{End}(E))$, we get:

$$\mathrm{d}(\mathrm{tr}(F_A^m)) = \mathrm{tr}\left(\mathrm{d}_{A^*\otimes A}(F_A^m)\right).$$

But then by the Leibniz property of connexions, applying it to the connexion $A^* \otimes A$, we can see that $\mathrm{d}_{A^*\otimes A}(F_A^m)$ is built out of terms which all include $\mathrm{d}_{A^*\otimes A}(F_A)$, which is $\equiv 0$ by 2nd Bianchi Identity (Lemma 3.5), and so the trace is zero. So hence done. [**Exercise** to check details.]

Longer Proof: If $\omega, \eta \in \Omega^*(\mathrm{End}(E))$, we can combine $\wedge$ with the commutator of endomorphisms to define the commutator $[\omega, \eta]$. We can then check:

- $[\omega, \eta] = -(-1)^{|\omega||\eta|}[\eta, \omega]$ (the first -1 is from the commutator, and the $(-1)^{|\omega||\eta|}$ is from the wedge product commuting).
- $[[\omega, \eta], \varphi] = [[\omega, \varphi], \eta] + (-1)^{|\omega||\varphi|}[\omega, [\eta, \varphi]]$.

These two properties tell us that $(\Omega^2(\mathrm{End}(E)), [\cdot, \cdot])$ forms a **super Lie algebra**.

Also, $\mathrm{d}[\omega, \eta] = [\mathrm{d}\omega, \eta] + (-1)^{|\omega|}[\omega, \mathrm{d}\eta]$.

Recall now that $(F_A)_\alpha = \mathrm{d}\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = \mathrm{d}\theta_\alpha + \frac{1}{2}[\theta_\alpha, \theta_\alpha]$, for θ_α the local connexion matrix, as the index $|\theta_\alpha| = 1$. So hence:

$$(\mathrm{d}F_A)_\alpha = \frac{1}{2}([\mathrm{d}\theta_\alpha, \theta_\alpha] - [\theta_\alpha, \mathrm{d}\theta_\alpha]) = [\mathrm{d}\theta_\alpha, \theta_\alpha],$$

by the super Lie algebra properties. Hence using the local expression of F_A, we have:

$$[\mathrm{d}\theta_\alpha, \theta_\alpha] = [(F_A)_\alpha, \theta_\alpha] - \frac{1}{2}\underbrace{[[\theta_\alpha, \theta_\alpha], \theta_\alpha]}_{=0 \text{ by super Lie algebra properties}}.$$

So we have: $(\mathrm{d}F_A)_\alpha = [(F_A)_\alpha, \theta_\alpha]$. So now we see (dropping the subscript, α, the trivialising chart, for notational simplicity)

$$\mathrm{d}(\mathrm{tr}(F_A \wedge \cdots \wedge F_A)) = \mathrm{tr}(\mathrm{d}F_A \wedge F_A \wedge \cdots \wedge F_A) + \mathrm{tr}(F_A \wedge \mathrm{d}F_A \wedge \cdots \wedge F_A) + \text{similar terms}$$

$$= \operatorname{tr}([F_A, \theta_A] \wedge F_A \wedge \cdots \wedge F_A) + \operatorname{tr}(F_A \wedge [F_A, \theta_A] \wedge \cdots \wedge F_A) + \text{similar terms}$$

where we have used the above. Then observe, if we take **any** conjugate-invariant function φ (like the trace, for example) on $\operatorname{End}(E)^{\otimes k}$, then for fixed X we have,

$$\varphi([X, X_1], X_2, \dots, X_k) + \varphi(X_1, [X, X_2], \dots, X_k) + \cdots + \varphi(X_1, \dots, X_{k-1}, [X, X_k]) = 0.^{\text{(ii)}}$$

So hence applying this with with $\varphi =$ trace, and $X = \theta_A$, $X_i = F_A$ for all i, gives the result from the above.

□

Proof of Lemma 3.8. Recall that we know from a previous calculation (Lemma 3.2 (ii)) that the difference of two connexions is a $C^\infty(M)$-module endomorphism-valued 1-form, and that $\mathbb{A}_E$, the space of connexions in E, is an affine space for $\Omega^1(\operatorname{End}(E))$.

So given d_A and d_B connexions in E, set:

$$\mathrm{d}_t = t\mathrm{d}_A + (1-t)\mathrm{d}_B \quad \text{for } t \in [0,1]$$

and set $F_t = \mathrm{d}_t^2$ (i.e. convex combination). We just need to show that $\operatorname{tr}(F_A^m) - \operatorname{tr}(F_B^m)$ is exact. So note, by the fundamental theorem of calculus/Stoke's theorem on $[0,1]$,

$$\operatorname{tr}(F_A^m) - \operatorname{tr}(F_B^m) = \int_0^1 \frac{\partial}{\partial t} \operatorname{tr}(F_t^m) \mathrm{d}t = \int_0^1 \operatorname{tr}(F_t^{m-1} \cdot \frac{\partial F_t}{\partial t}) \cdot m \, \mathrm{d}t$$

by commuting the derivative with the trace (easy to check by the definition of the trace of an endomorphism, as F_t is a polynomial in t). Now,

$$\frac{\partial F_t}{\partial t}(s) = \frac{\partial}{\partial t}(\mathrm{d}_t \circ \mathrm{d}_t)(s) = \frac{\partial(\mathrm{d}_t)}{\partial t} \cdot \mathrm{d}_t(s) + \mathrm{d}t \cdot \frac{\partial(\mathrm{d}_t)}{\partial t}(s) = L \cdot \mathrm{d}_t(s) + \mathrm{d}_t \cdot L(s),$$

where $L = \mathrm{d}_A - \mathrm{d}_B$ (from differentiating d_t with respect to the real parameter t). Now by Leibniz for the connexion d_t,

$$\mathrm{d}_t(L(s)) = (\mathrm{d}_t L)(s) - L(\mathrm{d}_t(s))$$

and so hence we see

$$\frac{\partial F_t}{\partial t} = \mathrm{d}_t L.$$

So,

$$\begin{aligned}
\operatorname{tr}(F_A^m) - \operatorname{tr}(F_B^m) &= \int_0^1 m \cdot \operatorname{tr}(F_t^{m-1} \cdot \frac{\partial F_t}{\partial t}) \, \mathrm{d}t \\
&= m \int_0^1 \operatorname{tr}\left((\mathrm{d}_t L) F_t^{m-1}\right) \, \mathrm{d}t \\
&= m \int_0^1 \operatorname{tr}\left(\mathrm{d}_t (L \cdot F_t^{m-1})\right) \, \mathrm{d}t
\end{aligned}$$

where we have used in the last line the Leibniz rule for d_t and the 2nd Bianchi identity to show $\mathrm{d}_t(F_t^{m-1}) = 0$ (as $\mathrm{d}_t F_t = \mathrm{d}_t^3$). Then using the same argument in the proof of Lemma 3.7, we have

(ii) This is simply because the expression on the LHS is simply:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(e^{tX} X_1 e^{-tX}, \dots, e^{tX} X_k e^{-tX})$$

and when φ is conjugation invariant, we have $\varphi(e^{tX} X_1 e^{-tX}, \dots, e^{tX} X_k e^{-tX}) = \varphi(X_1, \dots, X_k)$ is independent of t, and thus the derivative is zero.

$\mathrm{d} \circ \mathrm{Trace} = \mathrm{Trace} \circ \mathrm{d}_t$ (i.e. can commute a connexion with trace but then get exterior derivative instead), we have this

$$= m \int_0^1 \mathrm{d}\left(\mathrm{tr}(LF_t^{m-1})\right) \, \mathrm{d}t = \mathrm{d}\left(m \int_0^1 \mathrm{tr}(LF_t^{m-1}) \, \mathrm{d}t \right)$$

i.e. this is exact. Hence we have $[\mathrm{tr}(F_A^m)] = [\mathrm{tr}(F_B^m)]$ in $\mathrm{H}^{2m}_{\mathrm{dR}}(M)$, i.e. this class is independent on the connexion, as required.

□

Now we give an example of all of this.

Example: (**Sketched, Non-Examinable, Inappropriate** - See Complex Manifolds Part III)

There is an obvious notion of a complex vector bundle, with transition matrices/cocycle data $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{C})$.

A Hermitian metric on such a bundle is a family of Hermitian metrics on fibres, $\langle \cdot, \cdot \rangle_p$, varying smoothly (it can be shown that these always exist).

If $\{e_i\}_i$ is a local basis of sections of the complex vector bundle, and $\langle e_i, e_j \rangle = h_{ij}$, then this gives the local matrix H_α encoding the metric, i.e. if $s = \sum_i s_i e_i$ and $t = \sum_j t_j e_j$ are two local sections, then we have

$$\langle s, t \rangle_\alpha = \overline{t}_\alpha^T H_\alpha s_\alpha$$

where s_α, t_α are the vectors of coefficients. The transformation law of H can be found, since:

$$\overline{t}_\alpha^T H_\alpha s_\alpha = \langle s, t \rangle_\alpha = \langle s, t \rangle_\beta = \overline{t}_\beta^T H_\beta s_\beta$$

where $s_\alpha = \psi_{\alpha\beta} s_\beta$, etc, and so we get

$$H_\beta = \overline{\psi}_{\alpha\beta}^T H_\alpha \psi_{\alpha\beta}.$$

We now introduce $(T^*X)_{\mathbb{C}}$, the **complexification** of T^*X, where X is the base manifold and $E \to X$. So hence we have $\mathbb{C}$-valued differential forms (which locally look like $\mathrm{d}f + i\mathrm{d}g$ for some f, g).

A connexion on E here is: $\mathrm{d}_A : \Gamma(E) \to \Gamma(E \otimes (T^*X)_{\mathbb{C}})$, which is $\mathbb{C}$-linear with a Leibniz rule.

Suppose further that X itself was a complex manifold, i.e. locally $\cong \mathbb{C}^n$ with holomorphic transition maps. Then we can split $\mathbb{C}$-valued 1-forms into two types: ones which locally look like $\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j$, and ones which locally look like $\mathrm{d}\bar{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j$. So hence we see:

$$(T^*X)_{\mathbb{C}} \cong (T^*X)^{1,0} \oplus (T^*X)^{0,1}$$

where $(T^*X)^{1,0}$ has those locally like $\mathrm{d}z_i$ and $(T^*X)^{0,1}$ has those which locally look like $\mathrm{d}\bar{z}_j$. Then we have:

Fact: (Chern Connexion) On a complex bundle which is holomorphic over a complex manifold, $\exists!$ connexion (and so in particular, there is one) such that:

(i) It is unity, meaning:

$$\mathrm{d}(\langle s,t\rangle) = \langle \mathrm{d}_A s, t\rangle + \langle s, \mathrm{d}_A t\rangle$$

with respect to the chosen Hermitian metric,

(ii) In a local holomorphic basis of sections of E, we have: $\theta_\alpha = \theta_\alpha^{1,0}$, i.e. $\theta_\alpha^{0,1}$ (i.e. no antiholomorphic part in the connexion matrix, i.e. no $\mathrm{d}\bar{z}_j$).

Explicitly, we set: $\theta_\alpha = \theta_\alpha^{1,0} = H^{-1}(\partial H)$, where ∂ is the (1,0-part of the exterior derivative $\mathrm{d}: \Omega^0 \to \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ (this is differentiation with respect to z, not $\bar{z}$, in some sense).

Then one can check [**Exercise**] that this Chern connexion satisfies the transformation law to be a connexion.

So as an example, recall thinking about $\mathbb{C}P^1$ as: $\mathbb{C}P^1 = \mathbb{C}_1 \cup \mathbb{C}_2$, where $\mathbb{C}_1$ has coordinates z, and $\mathbb{C}_2$ has coordinates w, with transition function $w = 1/z$. Then we can specify a (complex) line bundle by its cocycle, $\psi_{12}: \mathbb{C}^* \to \mathrm{GL}_1(\mathbb{C})$, sending $z \mapsto 1/z$.

[**Exercise:** Show that this is really a tautological line bundle over $\mathbb{C}P^1$, i.e. $L \to \mathbb{C}P^1$ is such that the fibres are $L_z = [z] \subset \mathbb{C}^2$, where $[z]$ is the line generated by z.]

So to define a Hermitian metric on this complex vector (line) bundle, we need Hermitian matrices defining the metric locally, which here means on $\mathbb{C}_1$ and $\mathbb{C}_2$ (as this is our trivialising cover), obeying the transformation law found previously.

So here we need H_1 on $L_{\mathbb{C}_1}$ and H_2 on $L_{\mathbb{C}_2}$ such that: $H_2 = frac1\bar{z} \cdot H_1 \cdot \frac{1}{z}$, i.e. $H_2 = \frac{1}{|z|^2}H_1$. So then we can take:

$$H_1 = 1+|z|^2 \text{ on the } z\text{-chart } (\mathbb{C}_1) \quad \text{and} \quad H_2 = 1+|w|^2 \text{ on the } w\text{-chart } (\mathbb{C}_2)$$

and this works (as $w = \frac{1}{z}$ here).

Then if A is a connexion on L, viewed an an operator $\Gamma(L) \to \Gamma(L \otimes T^*(\mathbb{C}P^1))$, where these are $\mathbb{C}$-valued differential forms), then its curvature satisfies:

$$F_{A+a} = F_A + \mathrm{d}_{A^*\otimes A}(a) + \underbrace{a \wedge a}_{=0}$$

by some of our old analysis, i.e. we have $F_{A+a} = F_A + \mathrm{d}a$, for $a \in \Omega^1(\mathrm{End}(L))$, since the connexion $A^* \otimes A$ on $\mathrm{End}(L) \cong \mathbb{C}$ is the trivial line bundle, and thus is the trivial connexion (and so equals d). So hence in fact here, we see that the class $[F_A]$ is independent of A in $\mathrm{H}^2_{\mathrm{dR}}(X)$ (as if $[F_{A+a}] = [F_A]$ in $\mathrm{H}^2_{\mathrm{dR}}(X)$).

So now recall that we can define a connexion here by specifying that the connexion matrices are: $\theta_a = H_\alpha^{-1}\partial H_\alpha (\equiv \theta_\alpha^{1,0})$, where $\mathrm{d} = \partial + \overline{\partial} = \partial_z + \partial_{\bar{z}}$ with respect to coordinates $z = x+iy, \bar{z} = x - iy$.

So here, from the above we have

$$\theta_1 = H_1^{-1}\partial H_1 = \frac{1}{1+|z|^2} \cdot \frac{\partial}{\partial z}(1+|z|^2) = \frac{\bar{z}\mathrm{d}z}{1+|z|^2},$$

and so,

$$F_A = \mathrm{d}\theta_1 + \underbrace{\theta_1 \wedge \theta_1}_{=0 \text{ as 1-forms skew-commute}} = \mathrm{d}\theta_1 = (\partial + \bar{\partial})(\theta_1) = \frac{\partial}{\partial \bar{z}}\left(\frac{\bar{z}}{1+z\bar{z}}\right)\mathrm{d}\bar{z}\wedge \mathrm{d}z.$$

In polars, $\mathrm{d}\bar{z}\wedge\mathrm{d}z = -2ir\mathrm{d}r\wedge\mathrm{d}\theta$, and so this becomes

$$F_A = \left(\frac{1}{1+z\bar{z}} - \frac{\bar{z}z}{1+z\bar{z}}\right)(-2ir\mathrm{d}r\wedge\mathrm{d}\theta) = \frac{1}{(1+|z|^2)^2}(-2ir\mathrm{d}r\wedge\mathrm{d}\theta).$$

So hence,

$$\int_{\mathbb{C}P^1} F_A = \int_{\mathbb{C}_1} F_A = -i\int \frac{2r\mathrm{d}r\mathrm{d}\theta}{(1+r^2)^2} = -i\pi.$$

So note that as this does not equal 0, this confirms that $L \ncong \mathbb{C}$ is not the trivial line bundle.

Remark: If instead we took: $H_1 = (1+|z|^2)^n$, we would get a matrix on $L^{\otimes n}$ (with cocycle $\psi_{12}(z) = 1/z^2$), and similarly would find $\int_{\mathbb{C}P^1} F_A = -in\pi$.

3.2. **Torsion.**

If $E \to M$ is a vector bundle, and A is a connexion on E, then we know we have a map $\mathrm{d}_A : \Omega^0(E) \to \Omega^1(E)$ associated to the connexion. So now if we fix $X \in \Gamma(TM)$ a vector field, then we get a composition:

$$\Omega^0(E) = \Gamma(E) \xrightarrow{\mathrm{d}_A} \Gamma(E\otimes T^*M) \xrightarrow{\iota_X} \Gamma(E)$$

where ι_X is the interior product (i.e. the connexion gives an E-valued differential 1-form, which we then evaluatw at the vector field X to get something in E, so this gives another section).

Definition 3.5. *We call this operator $\iota_X \circ \mathrm{d}_A =: \nabla_X \equiv \nabla_X^A : \Omega^0(E) \to \Omega^0(E)$ the **covariant derivative of A along X**.*

Recall that d_A was called covariant differentiation with respect to the connexion A. Hence evaluation at X gives the covariant derivative along X.

Definition 3.6. *A connexion on TM is called an **affine** (or **kostul) connexion** on M.*

Given an affine connexion and a vector field X, we therefore get an operator

$$\nabla_X : \Gamma(TM) \to \Gamma(TM), \quad \text{via} \quad Y \longmapsto \nabla_X(Y)$$

from vector fields to vector fields. As seen before, an affine connexion A also induces a connexion d_{A^*} on the dual, i.e. T^*M, so we get:

$$\mathrm{d}_{A^*} : \underbrace{\Gamma(T^*M)}_{=\Omega^0(T^*M)} \to \underbrace{\Gamma(T^*M\otimes T^*M)}_{=\Omega^1(T^*M)}.$$

So hence we can consider the composition:

$$\underbrace{\Gamma(T^*M)}_{=\Omega^1(M)} \xrightarrow{d_{A^*}} \Gamma(T^*M \otimes T^*M) \xrightarrow{\pi} \underbrace{\Gamma(\Lambda^2(T^*M))}_{\Omega^2(M)}$$

where here $\Omega^i(M)$ are the differential i-forms on M, and we view $\Lambda^i V$ as a quotient of the tensor algebra by elements of the form $v \otimes v$: then the projection π is simply: $\pi(a \otimes b) = a \wedge b$, the wedge product.

So hence note, for $f \in C^\infty(M)$ and α a 1-form:

$$\pi(d_{A^*}(f \cdot \alpha)) = \pi(\alpha \otimes df + f \cdot d_{A^*}\alpha) = \alpha \wedge df + f(\pi \circ d_{A^*})(\alpha)$$

and

$$d(f \cdot \alpha) = df \wedge \alpha + f(d\alpha).$$

So hence we see:

$$(\pi \circ d_{A^*} + d)(f\alpha) = f(\pi d_{A^*} + d)(\alpha),$$

i.e. $\pi \circ d_{A^*} + d$ is a $C^\infty(M)$-module map $\Omega^1(M) \to \Omega^2(M)$.

Definition 3.7. *The* ***torsion*** τ_A *of an affine connexion A is the bundle homomorphism* $\tau_A = \pi d_{A^*} + d : \Omega^1(M) \to \Omega^2(M)$.

So hence we see that the torsion is a $C^\infty(M)$-module map.

Note: Unlike the curvature F_A of a connexion, the torsion is **only** defined for affine connexions.

So now we prove some equivalent expressions for the curvature and torsion of an affine connexion, in terms of the covariant derivative.

Proposition 3.1. *Let M be a manifold with an* ***affine*** *connexion A. Then:*

(i) The torsion τ_A is equivalent data to the map:
$$T : (X,Y) \longmapsto \nabla_X(Y) - \nabla_Y(X) - [X,Y].$$
(ii) The curvature (of this affine connexion) is equivalent data to the map:
$$K : (X,Y) \longmapsto \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

i.e. knowing one of T or τ_A determines the other, and same for F_A and K.

Note: Since $\tau_A \in \mathrm{Hom}(\Omega^1(M), \Omega^2(M))$, so fixing X, Y vector fields, we can contract out (i.e. compose with ι_X, ι_Y) the 2-form τ_A and get:

$$(\tau_A)(X,Y) \in \mathrm{Hom}(\Omega^1(M), \Omega^0(M)) = \Gamma(TM) \ni T(X,Y).$$

Also, we know $F_A \in \Omega^2(\mathrm{End}(TM))$, and so contracting against X, Y, we see that $F_A(X,Y) \in \mathrm{End}(TM)$, which is where $K(X,Y)$ lies.

Remark: For ease of memory, note that:

$$K(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

and something similar is true for T.

Proof. (i): Let $\alpha \in \Omega^1(M)$. Then we want to show:

$$\alpha(T(X,Y)) = \tau(\alpha)(X,Y)$$

which will prove the result, since then knowing one determines the other.

As always, we use the fact that:

$$d\alpha(X,Y) = X \cdot \alpha(Y) - Y\alpha(X) - \alpha([X,Y]).$$

Also, as $\nabla_X = \iota_X \circ d_A$, we see $\alpha(\nabla_X(Y)) = \alpha((\iota_X \circ d_A)(Y))$, and so noting that $d_A(Y) \in \Gamma(TM \otimes T^*M)$, and that α acts on the TM part of the tensor product, and ι_X acts on the T^*M part (by contraction), we see that as α and ι_X act on different parts of $TM \otimes T^*M$, the commute with each other here, and so:

$$\alpha(\nabla_X(Y)) = \iota_X(\alpha(d_A(Y))) = \alpha(d_A(Y))(X)$$

where we mean this in the only way that makes sense. Also, we know from the definition of the induced connexion d_{A^*} that:

$$d(\alpha(X)) = (d_{A^*}\alpha)(X) + \alpha(d_A(X)).$$

So now we just play around with these three expressions we have found:

$$\alpha(\nabla_X(Y)) = \alpha(d_A(Y))(X) = (d(\alpha(Y)) - (d_{A^*}\alpha)(Y))(X) = X \cdot \alpha(Y) - (d_{A^*}\alpha)(X,Y).$$

But then swapping $X \longleftrightarrow Y$, we get:

$$\alpha(\nabla_Y(X)) = Y \cdot \alpha(X) - (d_{A^*}\alpha)(Y,X).$$

So hence:

$$\begin{aligned}
\alpha(T(X,Y)) &= \alpha(\nabla_X(Y)) - \alpha(\nabla_Y(X)) - \alpha([X,Y]) \\
&= \underbrace{X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X,Y])}_{=d\alpha(X,Y)} + (d_{A^*}\alpha)(X,Y) - (d_{A^*}\alpha)(Y,X) \\
&= d\alpha(X,Y) + \pi(d_{A^*}\alpha(X,Y)) \\
&= (d\alpha + \pi d_{A^*}\alpha)(X,Y) \\
&= \tau_A(\alpha)(X,Y),
\end{aligned}$$

where we have used properties of π. So hence done.

(ii): Let $\{e_i\}_i$ be a local basis of sections of TM, and let $d_A e_i = \sum_j e_j \otimes \theta_{ji}$ be the usual local expression with the connexion matrix. So hence from the local expression for the curvature (or just applying d_A twice), we have:

$$F_A(e_i) = \sum_j \left(e_j \otimes d\theta_{ji} + \sum_k e_k \otimes (\theta_{kj} \wedge \theta_{ji}) \right).$$

So now we find what K looks like one this basis. So note:

$$\begin{aligned}\nabla_X \nabla_Y(e_i) &= \nabla_X((\iota_Y \circ \mathrm{d}_A)(e_i)) = \nabla_X\left(\iota_Y\left(\sum_j e_j \otimes \theta_{ji}\right)\right) = \nabla_X\left(\sum_j e_j \otimes \theta_{ji}(Y)\right)\\ &= \iota_X\left(\sum_j \mathrm{d}_A(e_j \otimes \theta_{ji}(Y))\right)\\ &= \iota_X\left(\sum_j \big(e_j \otimes \mathrm{d}(\theta_{ji}(Y)) + \theta_{ji}(Y)\cdot \mathrm{d}_A e_j\big)\right)\end{aligned}$$

where we have used the definition of ∇_X and the Leibniz rule for connexions, along with the local expression for $\mathrm{d}_A e_i$ above. So hence:

$$\nabla_X \nabla_Y(e_i) = \sum_j \left((X\cdot\theta_{ji}(Y))e_j + \theta_{ji}(Y)\sum_j \theta_{kj}(X)e_k\right).$$

Note that the same holds for $\nabla_Y\nabla_X(e_i)$ just by swapping $X \longleftrightarrow Y$. Then noting that $X\cdot\theta_{ji}(Y) - Y\theta_{ji}(X) = \mathrm{d}\theta_{ji}(X,Y) + \theta_{ji}([X,Y])$, we get:

$$\begin{aligned}\nabla_X\nabla_Y(e_i) - \nabla_Y\nabla_X(e_i) - \nabla_{[X,Y]}(e_i) &= \sum_j\left(\mathrm{d}\theta_{ji}(X,Y)e_j + \sum_k(\theta_{ji}(Y)\theta_{kj}(X) - \theta_{ji}(X)\theta_{kj}(Y))e_k\right)\\ &= F_A(e_i)(X,Y),\end{aligned}$$

by comparison with the above. So hence we have agreement on a basis, and so $K(X,Y) = F_A(X,Y)$, and so done.

□

4. Geometric Structures

4.1. **Affine Structures.**

Our aim here is to study some special geometric structures spaces can have (e.g. symplectic structures, Riemannian, etc) and what they imply/how they restrict the geometry, both locally and globally.

The **affine group** is the group of affine transformations of $\mathbb{R}^n$, i.e.

$$\mathrm{Aff}(\mathbb{R}^n) = \{M : \mathbb{R}^n \to \mathbb{R}^n : M(x) = Ax + b \text{ for some } A \in \mathrm{GL}_n(\mathbb{R}^n) \text{ and } b \in \mathbb{R}^n\}.$$

Here, A is the dilation and rotation, whilst b is the translation.

Definition 4.1. *An **affine structure** on a manifold is an atlas of charts such that the transition maps are restrictions of affine transformations (of some $\mathbb{R}^n$).*

Example: The n-torus T^n has a clear affine structure.

Theorem 4.1. *If M admits an affine connexion with zero curvature and zero torsion, then M admits an affine structure.*(iii)

Proof. Recall that $F_A \equiv 0 \Rightarrow \exists$ a local basis of covariant constant sections, say, $s_1, \dots, s_n \in \Gamma(TM)$.

So in this case, if A is our affine connexion, we can choose a local basis of sections such that $\mathrm{d}_A s_i = 0$. Then let ω_i be the dual basis of 1-forms to the s_i. So, $\omega_i \in \Gamma(T^*M)$ and $\omega_i(s_j) = \delta_{ij}$.

The from definition of the induced connexion A^*, we have:

$$(\mathrm{d}_{A^*}\omega_i)(s_j) = \underbrace{\mathrm{d}(\omega_i(s_j))}_{=\mathrm{d}(\text{constant})=0} - \omega_i(\underbrace{\mathrm{d}_A(s_j)}_{=0}) = 0.$$

So hence $\mathrm{d}_{A^*}\omega_i$ is zero on a basis, and so hence locally $\mathrm{d}_{A^*}\omega_i = 0$ for each i. Hence as the torsion is 0, and $\tau_A = \pi \mathrm{d}_{A^*} + \mathrm{d}$, we see that $\mathrm{d}\omega_i = 0$ for all i. So hence locally, since the ω_i are closed, locally they are exact (due to $\mathbb{R}^n$ having trivial de Rham cohomology groups for ranks ≥ 1). So hence we can write $\omega_i = \mathrm{d}x_i$ locally, for each i, for some $x_i \in C^\infty$.

Then since the ω_i are pointwise linearly independent (as they were a basis) we see that $\{x_1, \dots, x_n\}$ forms a local coordinate system on M. So now we have existence of such a coordinate system locally about each point of M, and so we just need to show that the transition maps between such coordinates are affine transformations.

So suppose another such coordinate system $y_1, \ldots, y_n$ gave local 1-forms $\mathrm{d}y_1, \ldots, \mathrm{d}y_n$, with $\mathrm{d}_{A^*})\mathrm{d}y_i) = 0$ (these are the properties of the coordinates system we just constructed above). Then on any overlap, we can write:

$$\mathrm{d}y_i = \sum_j B_{ji}\mathrm{d}x_j,$$

with the B_{ji} local smooth functions. So hence we have (by the Leibniz rule for connexions),

$$0 = \mathrm{d}_{A^*}(\mathrm{d}y_i) = \sum_j \mathrm{d}x_j \otimes \mathrm{d}B_{ji} + \sum_j B_{ji} \underbrace{\mathrm{d}_{A^*}(\mathrm{d}x_j)}_{=0}$$

$\Rightarrow \mathrm{d}B_{ji} = 0$, due to the linear independence of $\mathrm{d}x_j \otimes (\cdot)$. So hence the B_{ji} are constant functions. Moreover, this shows that

$$\mathrm{d}\left(y_i - \sum_j B_{ji}x_j\right) = 0$$

and so hence this is a constant function for each i (from definition of d on smooth functions) and so hence

$$y_i = \sum_j B_{ji}x_j + a_i$$

for some constant a_i, i.e. $\{y_1, \ldots, y_n\}$ differs from $\{x_1, \ldots, x_n\}$ by an affine transformation. So hence by taking these coordinates locally on M, we are done.

$\square$

Remarks:

(i) General topology ("the **developing map**") now gives in this case that $\exists$ a natural map $\tilde{M} \to \mathbb{R}^n$, where $\tilde{M}$ is the universal cover of M, and so M is the quotient of an open set in $\mathbb{R}^n$ by a discrete subgroup.

(ii) **Chern's Conjecture** ($\sim$ 1955): If M is a closed affine manifold, then $\chi(M) = 0$, where χ is the Euler characteristic.

(iii) **Bieberbach:** If in fact all the transition maps have no translation component (i.e. $b = 0$) then M has a finite cover which is a torus (see "Crystallographic Groups").

4.2. **Symplectic Structures.**

Moral: Some differential forms are 'more equal' than others!

Definition 4.2. *A **symplectic form** $\omega \in \Omega^2(M)$ on a manifold M is a 2-form such that:*

(i) ω is non-degenerate (as a skew-symmetric bilinear form on T_mM), i.e. if $\omega_x(u, v) = 0$ $\forall u \in T_xM$, then $v = 0$.

(ii) ω is closed, i.e. $\mathrm{d}\omega = 0$.

Note: If $u \in T_m M$, then we get a map $T_m M \to \mathbb{R}$ via: $\omega_m(u,\cdot)$, i.e. $v \mapsto \omega_m(u,v)$. So the non-degeneracy condition says that ω gives a natural isomorphism: $\omega : TM \xrightarrow{\cong} T^*M$, defined via: $(p,x) \mapsto \omega_p(x,\cdot)$ on fibres (i.e. $T_pM \to T_p^*M$).

Lemma 4.1 (Standard Form Theorem). *If V is a vector space which admits a non-degenerate skew-symmetric bilinear form ω, then:*

(i) $\dim_{\mathbb{R}}(V)$ *is even, and*

(ii) $\exists$ *a basis of V with respect to which ω acts as:*

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Proof. Let $u \in V$, $u \neq 0$. Then by non-degeneracy, $\exists v \in V$ such that $\omega(u,v) \neq 0$. So by renormalising (i.e. replace v by $v/\omega(u,v)$), we can assume wlog $\omega(u,v) = 1$. So on the space $W_1 = \mathrm{span}\langle u,v\rangle$, ω takes the form: $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ (as $\omega(u,u) = 0$ by skew-symmetry, and $\omega(u,v) = -\omega(v,u)$).

Then consider $W = W_1^{\perp\omega} = \{a \in V : \omega(a,u) = \omega(a,v) = 0\}$, the **orthogonal complement** of W_1 with respect to ω.

Then note that $\omega|_W$ is non-degenerate; indeed, if $a \in W\backslash\{0\}$, then we know by non-degeneracy of ω, $\exists b \in V$ such that $\omega(a,b) = 0$. But then as $\omega(a,u) = \omega(a,v) = 0$, b must have a non-trivial component in W (i.e. subtract off the components in the u and v direction, and this must give non-zero $\omega(a,\cdot)$ value). So considering this, we have $\omega(a,\tilde{b})$, and so $\omega|_W$ is non-degenerate.

But then as $\dim(W) = \dim(V) - 2$, we can induct on the dimension to conclude - note that $\dim(V)$ must be even, since otherwise we would end up with $\dim(W) = 1$, and so $W = \mathrm{span}\{u\}$ for some $u \neq 0$ with $\omega|_W$ non-degenerate. But then by skew-symmetry, $\omega(u,u) = 0$, and so ω would be non-degenerate, a contradiction. So done.

□

Note: A symplectic form ω on a vector space V is an element of $\Lambda^2(V^*)$.

Lemma 4.2. *If* $\dim_{\mathbb{R}}(V) = 2n$, *then*

$$\omega \text{ is non-degenerate} \iff \omega^n = \omega \wedge \cdots \wedge \omega \in \Lambda^{2n}(V^*) \text{ is non-zero.}$$

Proof. <u>($\Rightarrow$):</u> In the basis of the previous lemma, we have $\omega = e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2n-1} \wedge e_{2n}$, where $\{e_i\}_i$ is the dual basis of V of the previous lemma [**Exercise** to check]. Then hence $\omega^n = C \cdot e_1 \wedge \cdots \wedge e_{2n}$ for some $C \neq 0$ is immediate, and so $\omega^n \neq 0$.

<u>($\Leftarrow$):</u> **Exercise** to check. □

Corollary 4.1. *Let (M^{2n}, ω) be a closed, symplectic manifold (i.e. a manifold with a symplectic form). Then:*

(i) M is (naturally) oriented.

(ii) $H^{2i}_{dR}(M) \neq 0$ for all $0 \leq i \leq n$.

Proof. (i): Consider $\omega^n = \omega \wedge \cdots \wedge \omega$, which by the previous lemma, is a non-zero element of $\Omega^{2n}(M)$ since it is by definition non-degenerate, and so hence is a volume form.

(ii): We have $[\omega^n] \neq 0 \in \mathrm{H}^{2n}_{\mathrm{dR}}(M)$, since it is a volume form and so is closed, and if it were exact, then we would have a contradiction to Stoke's theorem.

But then we know that $[\omega^n] = [\omega]^n$, and so hence $[\omega]^i \in \mathrm{H}^{2i}_{\mathrm{dR}}(M)$ is a non-zero element in here for each i (it is closed since ω is closed, and if it were exact we would have a contradiction to $[\omega^n] \neq 0$ in $\mathrm{H}^{2n}_{\mathrm{dR}}(M)$, by Stoke's theorem).

$\square$

Example: Corollary 4.1 (ii) implies, for example, that S^{2n} admits no symplectic structure for $n \geq 2$ (e.g. S^4), since $\mathrm{H}^2_{\mathrm{dR}}(S^{2n}) = \mathrm{H}^2_{\mathrm{sing}}(S^{2n}, \mathbb{R}) = \{0\}$ for each such n.

Now let (V, ω) be a **symplectic vector space**, i.e. a vector space V with a symplectic form ω.

Definition 4.3. *A **Lagrangian subspace** $L \subseteq V$ of V a symplectic vector space is a half-dimensional subspace of V, i.e. $\dim_{\mathbb{R}}(L) = \frac{1}{2}\dim_{\mathbb{R}}(V)$, on which ω vanishes completely, i.e. $\omega|_{L\times L} \equiv 0$.*

In our normal form on V, we know we can take a basis so that

$$\omega = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{basis } (x_1, y_1)} \oplus \cdots \oplus \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{basis } (x_n, y_n)}.$$

So hence we see that $\mathrm{span}\langle x_1, \ldots, x_n\rangle$ and $\mathrm{span}\langle y_1, \ldots, y_n\rangle$ are Lagrangian subspaces of V (as all the x_i or y_i are in orthogonal parts).

Contrast this with $\mathrm{span}\langle x_1, y_1, \rangle$ being a **symplectic subspace** (i.e. a subspace is also symplectic, with the restriction of the symplectic structure on V given a symplectic form on this subspace).

Definition 4.4. *A submanifold $L \subseteq (M^{2n}, \omega)$ of a symplectic manifold is a **Lagrangian submanifold** if $\dim(L) = \frac{1}{2}\dim(M) = n$, and $\forall p \in L$, we have $T_pL \subset (T_pM, \omega_p)$ is a Lagrangian subspace (i.e. $i^*\omega = 0$, where $i : L \hookrightarrow M$ is the inclusion map).*

Note: For a symplectic manifold (M, ω), the tangent spaces (T_pM, ω_p) are symplectic vector spaces.

In fact, the cotangent bundle of a smooth manifold can **always** be given a symplectic structure so to make it a symplectic manifold, as we shall now see.

Example: [The cotangent bundle is always a symplectic manifold]

Let Q be any smooth manifold. Then the cotangent bundle T^*Q is canonically (always) a symplectic manifold.

Indeed, let $p : T^*Q \to Q$ be the canonical projection map. Then we claim that $\exists$ a tautological 1-form $\theta \in \Omega^1(T^*Q)$.

Let $M = T^*Q$ be the cotangent bundle. Let $X \in T_mM$. Then, define
$$\theta_m(X) = \xi(\mathrm{d}p_m(X)),$$
where $\mathrm{d}p_m : T_mM \to T_{p(m)}Q$ and $m \in T^*Q$. ξ is then determined by: $m = (p(m), \xi)$, i.e. $\xi \in T^*_{p(m)}Q$ is the 1-form part.

In local coordinates $x_1, \ldots, x_n$ on Q and $y_1, \ldots, y_n$ on the cotangent fibre near $m \in T^*Q$ (so $y_i = \mathrm{d}x_i$ is dual to $\frac{\partial}{\partial x_i}$, like position and momentum [this is how symplectic geometry relates to classical mechanics - see Symplectic Geometry Part III]), then:
$$p(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_n),$$
and so hence $\mathrm{d}p_m$ sends:
$$\mathrm{d}p_m\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} \quad \text{and} \quad \mathrm{d}p_m\left(\frac{\partial}{\partial y_i}\right) = 0$$
i.e. acts as the identity on the x_i, and sends rest to 0.

So if we write θ in terms of this local basis, so $\theta = \sum_i (a_i \mathrm{d}x_i + b_i \mathrm{d}y_i)$, then:
$$a_j = \theta\left(\frac{\partial}{\partial x_j}\right) = \xi\left(\frac{\partial}{\partial x_j}\right) = y_j$$
and similarly
$$b_j = \theta\left(\frac{\partial}{\partial y_j}\right) = \xi(0) = 0$$
just from the above. So hence we see,
$$\theta = \sum_{i=1}^n y_i \mathrm{d}x_i$$
in local coordinates, and so locally, we have
$$\mathrm{d}\theta = \sum_{i=1}^n \mathrm{d}y_i \wedge \mathrm{d}x_i$$
(as $\mathrm{d}^2 = 0$), which is symplectic form (indeed, it is pointwise the same as the standard symplectic form on $\mathbb{R}^{2n}$, from before). Hence $(T^*Q, \mathrm{d}\theta)$ (with the symplectic form defined locally) is a

symplectic manifold. [Note the symplectic form cannot be globally exact, as this would contradict some of our previous results on the de Rham cohomology].

In T^*Q, the zero-section $\cong Q \subset T^*Q$ is locally (in the above coordinates) $\{y = 0\}$, and so hence we see $\theta|_Q \equiv 0$, and $\omega = \mathrm{d}\theta|_Q = 0$ for the tautological 1-form above. [The **zero section** is just the section which sends $p \in Q$ to the zero point of the fibre at p.]

The cotangent fibres T_q^*Q of T^*Q are locally of the form $\{x_i = \text{constant } \forall i\}$, and so hence $\mathrm{d}x_i = 0$ on these fibres, and so:

$$\left(\sum y_i \mathrm{d}x_i\right)\Big|_{T_q^*Q} \equiv 0 \quad \text{and} \quad \omega|_{T_q^*Q} \equiv 0.$$

Hence these are all Lagrangian submanifolds - and so we see $\exists$ a family of Lagrangian manifolds sweeping/covering the entire space! (as the cotangent fibres cover all of T^*Q, and they are all Lagrangian submanifolds).

4.3. **Lagrangian Foliations.**

Definition 4.5. *A* ***polarisation*** *of a $2n$-dimensional symplectic manifold (M, ω) is a rank n subbundle of the tangent bundle $E \subset TM$ which is involutive and such that $\omega|_{E\times E} \equiv 0$,*

i.e. a polarisation is an involutive Lagrangian distribution. Note $\dim(E) = 2n$.

Example: In T^*Q as before, the subbundle $E \subset T(T^*Q)$ where $E_m = T_m(T_q^*Q)$, where $q = p(m)$ is the projection of m onto Q, is a polarisation of T^*Q.

□

So consider a polarisation E of a symplectic manifold (M^{2n}, ω). Then since E is involutive, the Frobenius integrability theorem (Theorem 1.3) implies that $\exists$ local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that locally, $E = \mathrm{span}\langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\rangle$.

The submanifolds $\{x = (x_1, \ldots, x_n) = (c_1, \ldots, c_n) : c \in \mathbb{R}^n$ is constant/fixed$\}$ for each given c are the local integrable submanifolds of E, and these are Lagrangian submanifolds of M (i.e. the submanifolds determined by constant x valued).

Theorem 4.2. *For E as above, $\exists$ a natural differential operator $\mathrm{d}_{E^*} : \Omega^0(E^*) \to \Omega^0(E^* \otimes E^*)$ which induces an affine connexion on the integrable submanifolds of E, which has vanishing curvature and torsion.*

[It will turn out that the connexion on E or E^* does not matter too much.]

Proof. We know that $\alpha_p(x) = \omega_p(x, \cdot)$ on tangent spaces T_pM induces an isomorphism $\alpha : TM \to T^*M$.

So define $\beta : TM \to E^*$ by, for each $p \in M$:

$$\beta_p(x) = \omega_p(x, \cdot)|_E,$$

i.e. $\beta(x) = \alpha(x)|_{E^*}$. Note that β is surjective, as α is an isomorphism. The kernel of β is clearly: $\ker(\beta) = \alpha^{-1}(E^0)$, where E^0 is the annihilator of E (i.e. those $x \in T_pM$ such that $\omega_p(x, e) = 0\ \forall e \in E$).

Note that clearly $E \subset \alpha^{-1}(E^0)$ (as E is a polarisation), and these are both subbundles of rank n ($\alpha^{-1}(E^0)$ has rank n because E^0 is n-dimensional, as E is a dimension n subspace of a $2n$ dimensional space, and α is an isomorphism). So hence we must have $E = \alpha^{-1}(E^0)$, i.e. $E = \ker(\beta)$.

So now let $\xi \in \Omega^0(E^*)$. Then we define a connexion $\mathrm{d}_{E^*}(\xi)$ via defining $\nabla_X(\xi)$ for $X \in \Omega^0(E)$, which we do via:

$$\nabla_X(\xi) := \beta([X, \tilde{\xi}]),$$

where $\tilde{\xi}$ satisfies $\beta(\tilde{\xi}) = \xi$ (such an $\tilde{\xi}$ exists as β is surjective), i.e. $\tilde{\xi}$ is a lift of ξ. Note that we can then define $\mathrm{d}_{E^*}(\xi)$ by requiring the usual relation, $\nabla_X = \iota_X \circ \mathrm{d}_{E^*}$.

Note that ∇_X is well-defined, since if $\beta(\tilde{\xi}_1) = \beta(\tilde{\xi}_2)$, then we have by linearity of β, $\tilde{\xi}_1\tilde{\xi}_2 \in \ker(\beta) = E$, and so since E is involutive, $[X, \tilde{\xi}_1 - \tilde{\xi}_2] \in \Omega^0(E)$, which implies, since $\ker(\beta) = E$,

$$\beta([X, \tilde{\xi}_1 - \tilde{\xi}_2]) = 0$$

i.e. $\beta([X, \tilde{\xi}_1]) = \beta([X, \tilde{\xi}_2])$. So this is well-defined.

Now as usual, if $X, Y \in \Omega^0(E)$, we define:

$$\nabla_X(Y) := \iota_X(\mathrm{d}_E(Y)).$$

where hence we can determine $\nabla_X(Y)$, and so hence $\mathrm{d}_E(Y)$, from the following relation:

$$X \cdot (\xi(Y)) = (\nabla_X \xi)(Y) + \xi(\nabla_X(Y))$$

for $\xi \in \Omega^0(E^*)$ (i.e. we define $\nabla_X(\xi)$ for such ξ,X as before, via β. This then define d_{E^*}. Then using the relation, we can define $\nabla_X(Y)$ for $Y \in \Omega^0(E)$ instead of $\xi \in \Omega^0(E^*)$, which then in turn allows us to define d_E).

> **Claim:** d_E and d_{E^*} as defined above are both connexions on the tangent and cotangent bundles to the integrable submanifolds Σ of E.
>
> *Proof of Claim.* Since $T_p(\Sigma) = E_p$ for all $p \in \Sigma$, Σ is an integrable manifold (this is because E is involutive). So as these maps are clearly linear by construction, we need to check the Leibniz condition of connexions (we will check this for d_{E^*}, and then its defining equation will give the result for d_E).
>
> So we want to consider $\mathrm{d}_{E^*}(f\xi)$ for $f \in C^\infty$ and $\xi \in \Omega^0(E^*)$. So suffices to check this on $X, Y \in \Omega^0(E)$, i.e.
>
> $$\mathrm{d}_{E^*}(f\xi)(X, Y) = \iota_X(\mathrm{d}_{E^*}(f\xi))(Y) = \nabla_X(f\xi)(Y) = \beta([X, f\tilde{\xi}])(Y),$$

where f is unaffected by the lift as β is linear and f is constant (i.e. $f(p) \in \mathbb{R}$) on each fibre. So recall as always,

$$[fX, gY] = fg[X,Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

So hence,

$$\begin{aligned} \mathrm{d}_{E^*}(f\xi)(X,Y) &= \big(\beta(f[X,\tilde{\xi}]) + (X \cdot f)(\beta(\tilde{\xi}))\big)(Y) \\ &= (f\nabla_X(\xi) + (X \cdot f)(\xi))(Y) \\ &= (f\mathrm{d}_{E^*}(\xi) + \xi \mathrm{d}f)(X,Y), \end{aligned}$$

where one again we have used the fact that β commutes with smooth functions (as they are constants on fibres). Hence as this is true for all X,Y, this gives the Leibniz connexion condition, so done.

□

So these are connexions. So hence we just need to proof the curvature and torsion vanish.

Claim: d_{E^*} has vanishing curvature.

[Note that this is true $\iff$ d_E has vanishing curvature, since $F_A \in \Omega^2(\mathrm{End}(G)) = \Omega^2(\mathrm{End}(G^*))$ for any bundle G with connexion A.]

Proof of Claim. We can consider: $\nabla_X(\nabla_Y\xi)$. By definition of $\nabla_X\xi$ for d_{E^*}, we have

$$\nabla_X(\nabla_Y\xi) = \beta([X, \widetilde{\nabla_Y\xi}]),$$

where we have $\beta(\widetilde{\nabla_Y\xi}) = \nabla_Y\xi$. But then again, we have $\nabla_Y\xi = \beta([Y,\tilde{\xi}])$, and so, $\beta(\widetilde{\nabla_Y\xi}) = \beta([Y,\tilde{\xi}])$. So hence as the definition of $\nabla_X\xi$ was independent of the choice of lift, and this shows that $[Y,\tilde{\xi}]$ is a lift of $\nabla_Y\xi$ under β, we may as well choose this as our lift(iv). Hence:

$$\nabla_X\nabla_Y(\xi) = \beta([X,[Y,\tilde{\xi}]]).$$

So by symmetry, we also have

$$\nabla_Y\nabla_X(\xi) = \beta([Y,[X,\tilde{\xi}]]).$$

Also, by definition of $\nabla_Z\xi$ again, we have

$$\nabla_{[X,Y]}(\xi) = \beta([[X,Y],\tilde{\xi}]).$$

So hence combining all of this, we have, by linearity of β,

$$\begin{aligned} \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi &= \beta\left([X,[Y,\tilde{\xi}]] + [Y,[\tilde{\xi},X]] + [\tilde{\xi},[X,Y]]\right) \\ &= \beta(0) = 0 \end{aligned}$$

by the Jacobi identity for the commutator/Lie bracket. Hence as $\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ directly determines the curvature (Proposition 3.1), this vanishing tells us that the curvature of d_{E^*} vanishes.

□

So now we just need to show that the torsions vanish.

Claim: d_E has vanishing torsion.

(iv) i.e. $\widetilde{\nabla_Y\xi}$ is defined to be anything that β maps to $\nabla_Y\xi$. So we can take it to be $[Y,\tilde{\xi}]$.

Proof of Claim. We saw before that $X \cdot \xi(Y) = (\nabla_X(\xi))(Y) + \xi(\nabla_X Y)$ (and this defined $\nabla_X(Y)$ for $X, Y \in \Omega^0(E)$). So hence by symmetry, we also have (swapping X and Y), $Y \cdot \xi(X) = (\nabla_Y \xi)(X) + \xi(\nabla_Y X)$.

So subtracting these expressions gives:

$$\begin{aligned}\xi(\nabla_X Y - \nabla_Y X) &= X \cdot \xi(Y) - Y \cdot \xi(X) - (\nabla_X \xi)(Y) + (\nabla_Y \xi)(X) \\ &= X \cdot \xi(Y) - Y \cdot \xi(X) - \beta([X, \tilde{\xi}])(Y) + \beta([Y, \tilde{\xi}])(X)\end{aligned}$$

where we have used the definition of $\nabla_X \xi$. But then by definition of β, this is simply saying:

$$\xi(\nabla_X Y - \nabla_Y X) = X \cdot \xi(Y) - Y \cdot \xi(X) - \omega([X, \tilde{\xi}], Y) + \omega([Y, \tilde{\xi}], X).$$

Also, from properties of the Lie derivative[v], we have:

$$0 = \mathfrak{L}_{\tilde{\xi}}(\underbrace{\omega(X, Y)}_{=0 \text{ as } E \text{ is Lagrangian, so } \omega|_{E\times E}=0}) = (\mathfrak{L}_{\tilde{\xi}}\omega)(X, Y) + \omega(\mathfrak{L}_{\tilde{\xi}}(X), Y) + \omega(X, \mathfrak{L}_{\tilde{\xi}}(Y)).$$

Then recall that for vector fields, we have $\mathfrak{L}_{\tilde{\xi}}(X) = [\tilde{\xi}, X]$, and so also $\mathfrak{L}_{\tilde{\xi}}(Y) = [\tilde{\xi}, Y]$. Hence we get:

$$(\mathfrak{L}_{\tilde{\xi}}\omega)(X, Y) = -\omega(\mathfrak{L}_{\tilde{\xi}}(X), Y) - \omega(X, \mathfrak{L}_{\tilde{\xi}}(Y)) = -\omega([\tilde{\xi}, X], Y) - \omega(X, [\tilde{\xi}, Y]).$$

So hence combining with the above, we have:

$$\xi(\nabla_X Y - \nabla_Y X) = X \cdot \xi(Y) - Y \cdot \xi(X) - (\mathfrak{L}_{\tilde{\xi}}\omega)(X, Y).$$

Then by Cartan's magic formula, we get:

$$\mathfrak{L}_{\tilde{\xi}}\omega = \iota_{\tilde{\xi}}(\underbrace{\mathrm{d}\omega}_{=0 \text{ as a symplectic form}}) + \mathrm{d}(\iota_{\tilde{\xi}}\omega).$$

Then note that: $\iota_{\tilde{\xi}}(\omega) = \omega(\tilde{\xi}, \cdot) = \beta(\tilde{\xi}) = \xi$. So hence this gives $\mathfrak{L}_{\tilde{\xi}}\omega = \mathrm{d}\xi$. So we get:

$$\begin{aligned}\xi(\nabla_X Y - \nabla_Y X) &= X \cdot \xi(Y) - Y \cdot \xi(X) - \mathrm{d}\xi(X, Y) \\ &= \xi([X, Y]),\end{aligned}$$

from the usual identity for $\mathrm{d}\xi(X, Y)$. So hence:

$$\xi(\nabla_X Y - \nabla_Y X - [X, Y]) = 0.$$

So hence as ξ was arbitrary, this tells us that $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ for all X, Y. This in turn tells us that the connexion has vanishing torsion, by Proposition 3.1. So done.

□

Hence this proves everything that was claimed, so done.

□

[v]We can prove that: $\mathfrak{L}_\xi(\omega(X_1, \ldots, X_n)) = (\mathfrak{L}_\xi \omega)(X_1, \ldots, X_n) + \omega(L_\xi X_1, X_2, \ldots, X_n) + \cdots + \omega(X_1, \ldots, X_{n-1}, \mathfrak{L}_\xi X_n)$. This can be proven by induction, with the base case $n = 1$ being proven by a direct check, since we know $\mathfrak{L}_\xi(X) = [\xi, X]$, $\mathfrak{L}_\xi \omega = \mathrm{d}(\iota(X)) + \iota(\mathrm{d}\omega)$, and $\mathfrak{L}_\xi(f) = \xi \cdot f$.

Remark: So hence as we know (by Theorem 4.1) that "vanishing curvature and torsion of a connection $\Rightarrow$ affine", we see that by the above, the integrable submanifolds of a polarisation E are always affine manifolds.

Remark: A question on Example Sheet 3 studies a related situation where (M,ω) is **foliated** (i.e. covered by a family of Lagrangian tori), essentially corresponding to when E is abelian, and not just involutive [compare this with/see **integrable systems**].

4.4. **Riemannian Structures.**

Definition 4.6. *Let $E \to M$ be a smooth vector bundle. Then a **metric on $\boldsymbol{E}$** is a smoothly varying family of inner products on the fibres, i.e. so is given by $g \in \Gamma(E^* \otimes E^*)$, which is symmetric in the two arguments on each fibre, and fibrewise non-degenerate.*

*In the case that $E = TM$, then we say g is a **Riemannian metric** on M.*

Note: Locally on M with coordinates $\{x_i\}_i$, a Riemannian metric looks like:

$$g = \sum_{i,j} g_{ij} \mathrm{d}x_i \otimes \mathrm{d}x_j,$$

with the matrix $(g_{ij})_{ij}$ being symmetric and positive definite (as $\mathrm{d}x_i \otimes \mathrm{d}x_j$ are a basis of sections of $T^*M \otimes T^*M$ locally). So Riemannian metrics always exist locally.

Note: A symplectic form is a non-degenerate element $\omega \in \Gamma(\Lambda^2 T^*M)$. A Riemannian metric is a non-degenerate element $g \in \Gamma(S^2(T^*M))$, for $S^2(T^*M)$ the second symmetric power. So these are the two "nice" classes of 2-forms.

Remark: A choice of metric g defines an isomorphism $E \to E^*$ via $v \mapsto g(\cdot, v)$.

Definition 4.7. *A connexion A on E is **compatible** with a metric g on E if:*

$$\mathrm{d}_{A^*\otimes A^*}(g) = 0,$$

i.e. the induced connexion on $E^* \otimes E^*$, where g lies, has g covariant constant.

Note: Equivalently, via unpacking the definition of the connexion $\mathrm{d}_{A^*\otimes A^*}$, they are compatible if $\forall u, v \in \Gamma(E)$,

$$\mathrm{d}(g(u,v)) = g(\mathrm{d}_A u, v) + g(u, \mathrm{d}_A v) \in \Gamma(T^*M),$$

where $\mathrm{d}_A u \in \Gamma(E \otimes T^*M) = \Omega^1(E)$. Note that this is just saying that this metric obeys the usual product rule of the dot product, $\frac{\mathrm{d}}{\mathrm{d}x}\langle f, g\rangle$, where $g(\cdot,\cdot) = \langle \cdot, \cdot \rangle_g$. Note that for fixed u, v, $g(u,v) \in C^\infty(M)$, since evaluation at p is simply the value $g_p(u,v)$.

By evaluating at $X \in \Gamma(TM)$ this is just saying that metric compatibility for a Riemannian metric is equivalent to:

$$\iota_X(\mathrm{d}g(Y,Z)) = g(\iota_X \mathrm{d}_A(Y), Z) + g(Y, \iota_X \mathrm{d}_A Z) \quad \forall X, Y, Z \in \Gamma(TM).$$

Remark: It turns out that **every manifold M admits some Riemannian metric** - see Example sheet 3.

Theorem 4.3 (The Fundamental Theorem of Riemannian Geometry)**.** *Suppose (M,g) is a* ***Riemannian manifold*** *(i.e. manifold M with Riemannian metric g). Then, $\exists!$ affine connexion A with vanishing torsion which is compatible with g.*

This connexion is called the ***Levi-Civita (LC) connexion*** *of the Riemannian manifold (M,g), and is denoted A_{LC}.*

Remark: This result tells us that there is no analogue of Chern-Weil theory for torsion (instead of curvature, which Chern-Weil theory used). This is because every manifold admits some Riemannian metric, and so we can find some affine connexion with vanishing torsion, so its de Rham cohomology class (and all its powers) will be zero.

Remark: This result also tells us that every manifold admits some affine connexion.

Proof. We will give one proof of this fact now, and then afterwards we will outline a more computational proof, with explicit forms of certain quantities.

First we establish existence of some metric compatible connexion. Then we will show we can get vanishing torsion.

By Gram-Schmidt, locally we can find an orthogonal (with respect to g) basis of vector fields, say $X_1,\dots,X_n$. Then define a connexion d_A by: $\mathrm{d}_A(X_i) = 0$ for all i. Then since for metric compatibility we need:

$$\iota_X \mathrm{d}g(Y,Z) = g(\iota_X \mathrm{d}_A(Y),Z) + g(Y,\iota_X \mathrm{d}_A(Z)).$$

By orthogonality we know $\mathrm{d}g(X_i,X_j) = 0$ for all i,j (as $g(X_i,X_j) = \delta_{ij}$ is a constant map, i.e. $g(X_i,X_j) \in C^\infty(M)$ via $p \mapsto g_p(X_i,X_j)$) and we also know that the RHS of this must also vanish on basis elements by definition of our d_A. So hence d_A is metric compatible locally. Then by taking a partition of unity subordinate to a trivialising cover and taking the relevant combination of the d_A (i.e. sum over and weighted by the partition of unity), the result is a global metric compatible connexion, and so we know such a connexion exists.

Now we show that we can modify our metric compatible connnexion to make it torsion-free. We know that the space of connexions is an affine space for $\Omega^1(\mathrm{End}(TM))$, and so hence if $\tilde{A}$ is another connexion, then $\mathrm{d}_{\tilde{A}} = \mathrm{d}_A + a$ for some $a \in \Omega^1(\mathrm{End}(TM))$. If both are metric compatible then we would have:

$$g(\iota_X \mathrm{d}_{\tilde{A}} Y, Z) + g(Y, \iota_X \mathrm{d}_A Z) = \iota_X \mathrm{d}g(Y,Z) = g(\iota_X \mathrm{d}_A Y, Z) + g(Y, \iota_X \mathrm{d}_A Z)$$

which upon substitution of $\mathrm{d}_{\tilde{A}} = \mathrm{d}_A + a$ we get:

$$g(\iota_X a(Y), Z) + g(Y, \iota_X a(z)) = 0 \quad \forall X,Y,Z \in \Gamma(TM).$$

This exactly says that a is a skew-symmetric endomorphism with respect to g, i.e. $a \in \Omega^1(\mathrm{SkewEnd}(TM))$. Indeed working this argument backwards, we see that any such skew-symmetric endomorphism $\tilde{a}$ gives rise to another metric-compatible connexion via $\mathrm{d}_A + \tilde{a}$, for our given connexion A before.

So how to the torsions of such connexions relate? Well by a direct calculation we have:

$$\begin{aligned}\tau_{\tilde{A}}(X,Y) &= \iota_X \mathrm{d}_{\tilde{A}} Y - \iota_Y \mathrm{d}_{\tilde{A}} X - [X,Y]\\ &= (\iota_X \mathrm{d}_A Y - \iota_Y \mathrm{d}_A X - [X,Y]) + \iota_X a(Y) - \iota_Y a(X)\\ &= \tau_A(X,Y) + \iota_X a(Y) - \iota_Y a(X)\end{aligned}$$

i.e. the torsion changes by adding on $\iota_X a(Y) - \iota_Y a(X)$.

So define a map $\Omega^1(\mathrm{SkewEnd}(TM)) \to \Gamma(TM \otimes \Lambda^2 T^*M) \equiv \Omega^2(TM)$ via $a \mapsto \tilde{a}$, where $\tilde{a}(X,Y) := \iota_X a(Y) - \iota_Y a(X)$. This is then a linear map between vector spaces of the same dimension. We wish to show that it is a bijection, since then this means we can modify τ_A to whatever we want, including 0 (which would then prove the result).

So as it is a linear map between vector spaces of the same (finite) dimension it suffices to show that it is injective. So suppose $\tilde{a} = 0$. Then we would know that $\iota_X a(Y) = \iota_Y a(X)$ for all $X, Y \in \Gamma(TM)$. But then since a is skew—symmetric we have:

$$\begin{aligned}g(\iota_X a(Y), Z) &= -g(Y, \iota_X a(Z)) = -g(Y, \iota_Z a(X))\\ &= g(\iota_Z a(Y), X) = g(\iota_Y a(Z), X)\\ &= -g(Z, \iota_Y a(X)) = -g(Z, \iota_X a(Y))\\ &= -g(\iota_X a(Y), Z)\end{aligned}$$

where we have used the symmetry of g. Hence we see that $g(\iota_X a(Y), Z) = 0$ for all Z and thus by non-degeneracy of g this implies $\iota_X a(Y) = 0$. But this is true for all X and so $a(Y) = 0$, and this is true for all Y and so $a \equiv 0$. Hence this map is injective.

So we get that locally, we can modify A to make it have zero torsion. Then uniqueness implies that we must have agreement on overlaps on a cover, and so we can glue together the pieces to get global existence and uniqueness. So we are done.

□

Remarks: There was nothing special about guaranteeing the torsion was 0. The proof also shows that for any torsion $\alpha \in \Omega^2(M)$, $\exists!$ metric-compatible connexion with torsion α. Taking $\alpha = 0$ is purely for convenience.

However this proof gives no insight into what the LC-connexion actually looks like! We shall look into this now.

4.4.1. *Alternative Viewpoint of the Levi-Civita.*

Suppose A is an affine connexion which is compatible with the metric, and is torsion-free (i.e. Levi-Civita). So, $\mathrm{d}_A : \Omega^0(TM) \to \Omega^1(TM)$, and so for $X \in \Gamma(TM)$, as usual we get the covariant derivative $\nabla_X = \iota_X \circ \mathrm{d}_A : \Gamma(TM) \to \Gamma(TM)$.

Then metric compatibility says:

$$\mathrm{d}(g(Y,Z))(X) = g((\mathrm{d}_A(Y))(X), Z) + g(Y, (\mathrm{d}_A(Z))(X))$$

i.e.

$$X \cdot g(Y,Z) = g(\nabla_X(Y), Z) + g(Y, \nabla_X(Z)),$$

for all vector fields $X, Y, Z \in \Gamma(TM)$.

The torsion-free condition becomes (as the torsion is equivalent to $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$), by the non-degeneracy of g,

$$g(\nabla_X Y, Z) - g(\nabla_Y X, Z) = g([X,Y],Z)$$

for all X, Y, Z.

So hence we can compute:

$$\begin{aligned} &X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y) \\ = &[\underbrace{g(\nabla_X Y, Z)}_{(1)} + \underbrace{g(Y, \nabla_X Z)}_{(2)}] + [\underbrace{g(\nabla_X Z, Y)}_{(3)} + \underbrace{g(Z, \nabla_Y X)}_{(1)}] - [\underbrace{g(\nabla_Z X, Y)}_{(2)} + \underbrace{g(X, \nabla_Z Y)}_{(3)}] \\ = &\underbrace{2g(\nabla_X Y, Z) - g([X,Y],Z)}_{(1)} + \underbrace{g([X,Z],Y)}_{(2)} + \underbrace{g([Y,Z],X)}_{(3)}, \end{aligned}$$

where we have combined the terms (1) on the second line to get the corresponding term on the third line, etc, using the above relation for torsion-free. So rearranging, we have:

$$g(\nabla_X Y, Z) = \frac{1}{2}\left[X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\right].$$

So hence if a metric-compatible, torsion-free connexion exists, then its covariant derivative must be given by the above, as the RHS only depends on X, Y, Z and g. Hence varying Z means, by non-degeneracy of g, we can determine $\nabla_X Y$, and hence we can determine ∇_X for all such X. This in turn uniquely determines d_A.

So hence this **gives another proof of the uniqueness** of the LC-connexion.

Then to prove existence, one can take the above expression as a definition, and then check that it defines a connexion with the relevant properties, i.e. check that:

(i) $\nabla_{fX}(Y) = f\nabla_X(Y)$
(ii) $\nabla_X(fY) = (X \cdot f)(Y) + f\nabla_X(Y)$
(iii) $\nabla_X Y - \nabla_Y X - [X,Y] = 0$
(iv) $g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = X \cdot g(Y,Z)$

for all $f \in C^\infty(M)$, X, Y, Z vector fields. Note that (i)+(ii) are the connexion properties, (iii) is saying the connexion is torsion-free, and (iv) says that it is compatible with the metric g.

Checking these properties is not hard, but is tedious. However all of this does give an alternative proof of the existence theorem of the LC-connexion. To show properties (i)-(iv) here, properties such as $Y \cdot (f\varphi) = (Y \cdot f)\varphi + \varphi(Y \cdot f)$ for functions f and $\varphi = g(X,Z)$, and $[fX, gY] = fg[X,Y] + f(X \cdot g)Y - g(Y \cdot f)X$, etc, need to be used.

In local coordinates $x_1, \ldots, x_n$ of M, and hence a local basis of vector field $\partial_i = \frac{\partial}{\partial x_i}$ of TM, then writing $g_{ab} = g(\partial_a, \partial_b)$, and noting/recalling that $[\partial_i, \partial_j] = 0$ for all i, j, then the expression above gives:

$$g(\nabla_{\partial_i}(\partial_j), \partial_k) = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$$

by taking $X = \partial_i$, etc. So hence the LC-connexion is determined explicitly in terms of the metric coefficients, $(g_{ij})_{ij}$.

Proposition 4.1. *Let (M, g) be a Riemannian manifold. Then, if $F_{LC} \equiv 0$, then M is locally Euclidean,*

i.e. $\exists$ local coordinates $x_1, \ldots, x_n$ about each point p of M such that $g_p = \sum_{i=1}^n \mathrm{d}x_i \otimes \mathrm{d}x_i$.

Note: F_{LC} is the connexion of the LC-connexion. Note that by definition, $\tau_{LC} \equiv 0$, i.e. the torsion of the LC-connexion vanishes.

Proof. Compare this proof with the discussion in §4.1 on affine structures - we use a very similar argument here.

Since $F_{LC} \equiv 0$, then by Theorem 3.1, we know that $\exists$ a local basis of covariant constant sections for the LC-connexion. So $\exists$ $s_1, \ldots, s_n$ a local basis of $\Gamma(TM)$ with $\mathrm{d}_{LC}(s_i) = 0$, where d_{LC} is the LC-connexion.

Then as in the discussion of affine structures, we get local coordinates $\tilde{x}_1, \ldots, \tilde{x}_n$ (i.e. C^∞ functions locally) such that $\{\mathrm{d}\tilde{x}_i\}_i$ is the dual basis of $\{s_i\}_i$ (this is also since the torsion of the LC-connexion vanishes).

So we can write: $g = \sum_{ij} g_{ij} \mathrm{d}\tilde{x}_i \otimes \mathrm{d}\tilde{x}_j$ locally. Then the metric-compatibility of the LC-connexion implies:

$$\mathrm{d}_{LC^* \otimes LC^*}(g) = 0 \quad \text{(i.e. “}g\text{ is covariant constant”)}$$

i.e.

$$\mathrm{d}(g(u,v)) = g(\mathrm{d}_{LC} u, v) + g(u, \mathrm{d}_{LC} v)$$

for all vector fields u, v. So hence choosing $u = s_i$, $v = s_j$, this gives (as $\mathrm{d}\tilde{x}_i(s_j) = \delta_{ij}$ as dual basis)

$$\mathrm{d}g_{ij} = 0 \quad \forall i, j$$

i.e. $(g_{ij})_{ij}$ is a constant matrix (note that locally M is connected, as it looks like $\mathbb{R}^n$). But then we know that $(g_{ij})_{ij}$ is symmetric and positive definite always (as it is an inner product), and so hence here it is orthogonally diagonalisable, i.e $\exists$ P orthogonal such that $P(g_{ij})_{ij}P^T$ is diagonal. So if we set: $y_j = \sum_i P_{ij}\tilde{x}_i$ (i.e. change basis via P), then in the $\{y_j\}_j$ coordinates, g is in the required form. So done.

□

4.5. **Riemann Curvature.**

Let (M, g) be a Riemannian manifold. Then we know that it has a distinguished connexion, A_{LC}, the Levi-Civita connexion.

Definition 4.8. *The **Riemann curvature** of (M, g) is the curvature F_{LC} of A_{LC}.*

Note that $F_{LC} \in \Omega^2(\mathrm{End}(TM)) = \Gamma(TM \otimes T^*M \otimes \Lambda^2 T^*M)$.

Notation: For vector fields X, Y, we write $R(X,Y) \in \mathrm{End}(TM)$ for the operator:

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Note that from before (Proposition 3.1), $R(X,Y)$ directly determines the curvature F_{LC} (here, all covariant derivatives are for the LC connexion).

Note that if we have local coordinates $x_1, \ldots, x_n$ on M, and so hence $\partial_i := \frac{\partial}{\partial x_i}$ are a local basis of vector fields (i.e. of $\Gamma(TM)$), then, and then locally $R(X,Y)$ is a map $TM \to TM$ (i.e. is defined on some local part of TM, not on all of TM), since we can choose $\{\partial_i\}_i$ as a basis of TM, we can write:

$$R(\partial_k, \partial_l)(\partial_j) = \sum_i R^i_{jkl} \partial_i$$

for some coefficients R^i_{jkl}, called the **(1,3)-curvature**. In summation convention (where upper indices represent dual basis vectors) we would have

$$F_{LC} = R^i_{abc} e_i \otimes e^a \otimes e^b \otimes e^c$$

and so

$$F_{LC}(e_k, e_l) = R(e_k, e_l) = R^i_{abc} e_i \otimes e^a \cdot e^b(e_k) \cdot e^c(e_l) = R^i_{abc} e_i \otimes e^a \delta^b_k \delta^l_c = R^i_{akl} e_i \otimes e^a.$$

In particular, $R(e_k, e_l)(e_j) = R^i_{jkl} e_i$.

One can also define:

$$R(W,Z,X,Y) := g(R(X,Y)(Z), W)$$

which similarly defines a (0,4)-tensor by the coefficients on the basis vectors. Indeed, taking $W = \partial_i$, $Z = \partial_j$, etc, we have:

$$R_{ijkl} := R(\partial_i, \partial_j, \partial_k, \partial_l) = g(R(\partial_k, \partial_l)\partial_j, \partial_i) = g\left(\sum_m R^m_{jkl} \partial_m, \partial_i\right) = \sum_m g_{mi} R^m_{jkl},$$

where once again, $g_{mi} = g(\partial_m, \partial_i)$.

Remark: If instead of working relative to a basis $\{\partial_i\}_i$ of T_pM, we worked in an orthonormal basis $\{e_i\}_i$ of T_pM with respect to g, then $g_{ij} = \delta_{ij}$, and then by the above we see:

$$R_{ijkl} = R^i_{jkl}.$$

In such an orthonormal basis, we clearly have (from Proposition 3.1, how $R(X,Y)$ determines F_{LC})

$$F_{LC} = \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \cdot e_i \otimes e_j \otimes e_k \wedge e_l$$

where we have used $TM \overset{g}{\cong} T^*M$ to identify some basis vectors.

Symmetries: Now we list/prove a lot of symmetries about these quantities:

(i) $R_{ijkl} = -R_{ijlk}$, which is true by the skew-symmetry of the 2-form R, i.e. since $R(\partial_k, \partial_l) = -R(\partial_l, \partial_k)$ (i.e. since F_{LC} is anti-symmetric in last two components).

(ii) $R_{ijkl} = -R_{jikl}$, as the endomorphism of TM is skew, by metric compatibility.

(iii) $R_{ijkl} + R_{iklj} + R_{iljk} = 0$, by vanishing torsion. More conveniently this is often written as:

$$R_{i[jkl]} = 0$$

where $[jkl]$ denotes the permutation of these indices.

How do we see this? Well, since we know $[\partial_k, \partial_l] = 0$ for these coordinate vector fields, and since the torsion is determined by: $\nabla_X Y - \nabla_Y X - [X, Y]$, one then sees as this vanishes, that $\nabla_{\partial_k}(\partial_j) = \nabla_{\partial_j}(\partial_k)$. But then as:

$$R(\partial_k, \partial_l)(\partial_j) = \nabla_{\partial_k}\nabla_{\partial_l}(\partial_j) - \nabla_{\partial_l}\nabla_{\partial_k}(\partial_j) - \underbrace{\nabla_{[\partial_k,\partial_l]}}_{=0 \text{ as } [\partial_k,\partial_l]=0}(\partial_j)$$

we see by adding this up for the three cyclic permutations of k, l, j, all terms cancel out. So hence this shows:

$$R(X, Y)(Z) + R(Y, Z)(X) + R(Z, X)(Y) = 0$$

for all $X, Y, Z \in \Gamma(TM)$ vector fields, and so choosing $X = \partial_k$, etc, and evaluating at ∂_i gives the result.

(iv) $R_{ijkl} = R_{klij}$: this follows as an unenlightening exercise [**Exercise** to check!] by writing out symmetry (iii) for each of the possible orders (i.e. i fixed as first index, then j, etc, with all possible combinations of the others), and adding them, using symmetries (i) and (ii) to cancel everything possible.

(v) $\partial_i(R^m_{ljk}) + \partial_j(R^m_{lki}) + \partial_k(R^m_{lij}) = 0$: this comes from the 2nd Bianchi identity. Again more conveniently we often write this as:

$$R^m_{l[ij;k]} = 0$$

where the semi-colon represents a derivative. How do we see this? Recall that locally, $F_A = d\theta + \theta \wedge \theta$ in terms of the connexion matrix θ. Then, Bianchi 2nd says: $d_{A^*\otimes A}(F_A) = 0$, for any connexion A. So if we write $\Theta = d\theta + \theta \wedge \theta$ ($= F_A$ locally), then locally Bianchi 2nd says for the LC connexion (unravelling this induced connexion):

$$d\Theta = \Theta \wedge \theta - \theta \wedge \Theta = d\theta \wedge \theta - \theta \wedge d\theta.$$

Fact: On any Riemannian manifold (M, g), $\exists$ local coordinates at a point $p \in M$ such that the Christoffel symbols of A_{LC} all vanish at p (see Geodesic normal coordinates, §5).

[**Recall:** The **Christoffel symbols**, Γ^k_{ij}, are the coefficients of the connexion matrix, i.e. each component of the connexion matrix can be written as $\theta_{jk} = \sum_i \Gamma^k_{ij} dx_i$.]

But we know that we can write: $\Theta_{ij} = \sum_{k,l} R^i_{jkl} dx_k \wedge dx_l$, for a local expression for the curvature. So hence if $\theta = 0$ at p (can do by above fact), then $d\theta = 0$ at p, and so hence $d\Theta = 0$, i.e. (plus using symmetries (i) and (ii) above)

$$\sum_{k,l,m} \partial_m R^i_{jkl} \cdot dx_m \wedge dx_k \wedge dx_l.$$

So hence grouping together the same terms of $dx_m \wedge dx_k \wedge dx_l$ (by commutation properties of $\wedge$), we get this above relation.

Note: It is **crucial** to note that the final relation (v) above **only holds true** in geodesic normal coordinates. This means that the derivatives are taken with respect to these coordinates, and not any others.

Definition 4.9. *The* ***Ricci curvature*** *$Ric \in \Gamma(T^*M \otimes T^*M)$ is the trace:*

$$Ric(e_i, e_j) = \sum_k R_{kikj}$$

for $\{e_i\}_i$ a local orthonormal basis of T_pM for g.

Note: More generally, in a different, non-orthonormal basis of T_pM (where we use a local basis and take its dual basis, $\{\partial_i\}_i$, which is not necessarily orthogonal with respect to g) we have

$$\mathrm{Ric}(\partial_i, \partial_j) = \sum_l R^l_{ijl} = \sum_{l,m} g^{lm} R_{limj},$$

if $g^{lm} = (g_{lm})^{-1}$.

Note: The Ricci curvature and the Riemannian metric both lie in the same space, i.e. $\mathrm{Ric}, g \in \Gamma(T^*M \otimes T^*M)$.

Something important happens when they are $C^\infty(M)$ multiplies of one another:

Definition 4.10. *We say that the Riemannian manifold (M, g) is* ***Einstein*** *if $\exists \lambda \in C^\infty(M)$ such that:*

$$Ric = \lambda g.$$

Remark: In signature $(+,-,-,-)$, this is Einstein's equation from General Relativity!

Proposition 4.2. *If $\dim_{\mathbb{R}}(M) \neq 2$, and (M, g) is a connected Einstein manifold, then λ is constant (λ as in $Ric = \lambda g$).*

We call λ the ***cosmological constant****. So this result says that for essentially all Einstein manifolds, Ric is a constant multiple of g.*

Proof. Write for general a_{ijk}, simply for notational convenience, $a_{ijk,m} = \frac{\partial}{\partial x_m}(a_{ijk})$. Write $\partial_i = \frac{\partial}{\partial x_i}$ (here, $(x_1, \ldots, x_n)$ is a local basis of M). Then symmetry (v) above gives:

$$R^i_{jkl,m} + R^i_{jmk,l} + R^i_{jlm,k} = 0$$

for all i, j, k, l, m (here, $R^l_{ijk,m} = \partial_m(R^l_{ijk})$, etc). So setting $k = j$ in this and summing over j we get:

$$\sum_j (R^i_{jjl,m} + R^i_{jmj,l} + R^i_{jlm,j}) = 0.$$

So hence writing $R_{ij} := \mathrm{Ric}(e_i, e_j) = \sum_k R_{kikj}$ for this trace, in an orthonormal basis of TM with respect to g, we have $R^i_{jkl} = R_{ijkl}$, and so hence using the above with symmetries (i) and (ii), we get

$$-R_{il,m} + R_{im,l} + \sum_j R_{ijlm,j} = 0.$$

But then note as $R_{ij} = \text{Ric}(e_i, e_j) = \lambda g(e_i, e_j) = \lambda \delta_{ij}$ here, this gives (as δ_{ij} is a constant),

$$-\partial_m \lambda \delta_{il} + \partial_l \lambda \delta_{im} + \sum_j R_{ijlm,j} = 0.$$

So now set $i = l$ and sum over i to get:

$$-n\partial_m \lambda + \partial_m \lambda + \sum_j R_{jm,j} = 0.$$

But then again, $\sum_j R_{jm,j} = \sum_j \partial_j \lambda \delta_{jm} = \partial_m \lambda$, and so this becomes:

$$(2-n)\partial_m \lambda = 0,$$

and this holds for any m. So as $n \neq 2$ ($n = \dim_{\mathbb{R}}(M)$), we get $\frac{\partial}{\partial x_m}(\lambda) = 0$ for all m. So hence as M is connected, and λ is a smooth function on M, this tells us that λ must be constant.

□

Definition 4.11. *The **scalar curvature** is the trace of the Ricci curvature, i.e.*

$$S = \sum_{i=1}^{n} Ric(e_i, e_i).$$

Thus the scalar curvature is just a function on M, i.e. $S(p) =$ trace of Ricci curvature on T_pM. In particular, we see that connected Einstein manifolds of dimension $\neq 2$ have constant scalar curvature, since this is then just the trace of λg, which is constant ($= n\lambda$ in an orthonormal basis).

So how do we get Einstein manifolds? One way is to give us lots of symmetry, as in the next theorem:

Theorem 4.4. *Let (M, g) be a Riemannian manifold. Suppose that the Lie group $G \leq Isom(M, g)$ acts transitively on M via isometries. Then, fix $m \in M$ and let $H = Stab_G(m)$. Suppose H acts irreducibly on T_mM (i.e. only invariant subspaces are T_mM and $\{0\}$. Then, (M, g) is Einstein (with constant cosmological constant).*

Remark: If G acts on M and m is a fixed by g (i.e. $g \cdot m = m$ for all $g \in H \leq G$), then $\exists$ a natural representation of H via: $H \to \text{End}(T_mM)$. Indeed, if $\Phi : G \times M \to M$ and let let: $\varphi_h(m) = \Phi(h, m)$, then: $D\varphi_h|_m : T_mM \to T_{h(m)}M$. But then $h(m) = m$, and so $D\varphi_h|_m : T_mM \to T_mM$ is an endomorphism of T_mM.

Proof. Recall that by symmetry (iii), $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. So taking $i = k$ in this and summing over k gives:

$$\sum_k (R_{kjkl} + \underbrace{R_{kklj}}_{=0 \text{ by skew-symmetry}} + R_{kljk}) = 0$$

i.e.

$$\sum_k (R_{kjkl} - R_{klkj}) = 0$$

via flipping two of the indices. By then by definition of the Ricci curvature, as these are just traces this gives: $R_{jl} = R_{lj}$.[(vi)]

So hence if we define: $\theta : T_mM \to T_mM$ via: $g(\theta(v), w) := \text{Ric}(v, w)$, which determines $\theta(v)$ on a basis, etc (as g is a non-degenerate inner product), then we see that θ is self-adjoint as an endomorphism of (T_mM, g), by the symmetry of the Ricci curvature above (i.e. $g(\theta(v), w) = \text{Ric}(v, w) = \text{Ric}(w, v) = g(\theta(w), v) = g(v, \theta(w))$).

So hence as $H \leq G \leq \text{Isom}(M, g)$, this implies that H preserves the Ricci tensor (i.e. isometries preserve Riemann curvature).

$$\Rightarrow \text{ The } \theta \text{ eigenspaces are } H\text{-invariant,}$$

where θ is the above self-adjoint map. So hence as H acts irreducibly, this implies that there is only one eigenvalue, say λ, with eigenspace T_mM. So hence $\forall v, w \in T_mM$, we have:

$$\text{Ric}(v, w) = g(\theta(v), w) = g(\lambda v, w) = \lambda g(v, w),$$

i.e. $\text{Ric} = \lambda g$, on $T_mM \otimes T_mM$. Let λ_m denote this constant λ on T_mM, and so hence $\lambda(m) = \lambda_m$ is a function on M. Then since the G action is transitive, this means that $\forall \tilde{m} \in M$, $\exists g \in G$ such that $g \cdot m = \tilde{m}$, and so $\lambda_{\tilde{m}} = \lambda_{g\cdot m} = \lambda_m$ for all $\tilde{m}$, i.e. λ is constant on M (as the action is by isometries which preserve g and Ric).

So hence (M, g) has: $\text{Ric} = \lambda g$, for some constant λ, everywhere on M, and so hence (M, g) is Einstein.

□

Example 4.1. *We know $O(n+1)$ acts on $S^n \subset \mathbb{R}^{n+1}$. So equip S^n with the metric induced from $g_{Euclidean}$ (on $\mathbb{R}^{n+1}$ - see Example Sheet 3).*

Then, $Stab(e_1) \cong O(n)$, where $e_1 = (1, 0, \ldots, 0)$, and this acts naturally on $T_{e_1}S^n = \{(0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}\} \cong \mathbb{R}^n$. But as $O(n)$ acts irreducibly on $\mathbb{R}^n$, the Theorem 4.4 $\Rightarrow$ (S^n, g) is Einstein.

Example 4.2. *Let $M = O(n)$. Then we know that $T_eM = \mathfrak{g}\{$Skew-symmetric matrices$\}$ from a previous calculation.*

*So define $g_e(A, B) = -tr(AB)$, the trace of this product. This can be checked to be a positive definite bilinear form on $\mathfrak{g}$ [**Exercise** to check].*

Now define a metric g on $O(n)$ by insisting that it is left invariant, i.e. require:

$$\langle L_h\alpha, L_h\beta\rangle_h := \langle \alpha, \beta\rangle_e$$

where $\langle \cdot, \cdot\rangle_h$ is the inner product on the tangent space at h, and L_h is left-multiplication by h.

[(vi)] i.e. Ricci curvature is always symmetric. This is could since the inner product is always symmetric and so we need this is we hope to have $\text{Ric} = \lambda g$.

So since the trace is conjugation invariant, g is invariant under the action of left translation and conjugation by $O(n)$ on itself, and hence g is right invariant as well (so g is bi-invariant). So hence if we set: $G = O(n) \times O(n) \leq Isom(M, g)$, where the left and right parts of G acts on the left and right respectively (i.e. $(h, k) \in G$ acts by: $(h, k) \cdot m = hmk^{-1}$ in matrix multiplication), then we see that:

$$Stab_G(e) = \{(h, h) : h \in O(n)\}.$$

*Then [**Exercise**] taking $H = O(n)$, we can show that H acts irreducibly on $\mathfrak{g}$, and hence deduce that $(O(n), g)$ is Einstein.*

Example: [Non-Example, Sketch]

When studying Chern-Weil theory, we found characteristic classes in $\mathrm{H}^*_{\mathrm{dR}}(M)$ from the traces of powers of curvature. The **Euler class**, $e(M) \in \mathrm{H}^n_{\mathrm{dR}}(M)$, of a closed and oriented n-manifold M, is characterised by:

$$\int_M e(M) = \chi(M),$$

where χ is the Euler characteristic. It is obtained from Chern-Weil theory.

If M^4 is a closed, oriented 4-manifold, this can be used to show:

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(|R|^2 - |z|^2\right) \mathrm{vol}_g$$

where g is a choice of Riemannian metric, with vol_g the associated volume form on M. R, z are expressions in terms of F_{LC}, such that:

$$z = 0 \iff g \text{ is Einstein.}$$

So hence we see:

$$(M, g) \text{ is Einstein } \Rightarrow z = 0 \Rightarrow \chi(M) \geq 0.$$

So for example, $M = T^4$ has $\chi(M) < 0$, and so T^4 cannot be Einstein for any Riemannian metric.

5. Geodesics

Let $\gamma : (a, b) \to M$ be a smooth curve. Suppose that M has an affine connexion, denoted ∇^{aff} (note from before we know that M always has an affine connexion, via the LC-connexion). Then, γ^*TM gives a vector bundle over (a, b) via the pullback of the tangent bundle TM via γ. This leads to the idea of pulling back a connexion (known as a **pullback connexion**). We look at the example of geodesics.

Lemma 5.1. *$\exists$ an operator $\nabla_{\frac{\partial}{\partial t}} : \Omega^0(\gamma^*TM) \to \Omega^0(\gamma^*TM)$ (recall $\Omega^0(\gamma^*TM) = \Gamma(\gamma^*TM)$) such that:*

(i) We have the Leibniz property:

$$\nabla_{\frac{\partial}{\partial t}}(fY)\Big|_t = f'(t)\, Y|_t + f(t)\, \nabla_{\frac{\partial}{\partial t}}(Y)\Big|_t$$

for all $f \in C^\infty(a, b)$ and for all $Y \in \Gamma(\gamma^*TM)$ vector fields along γ.

(ii) If at $t_0 \in (a, b)$, $\exists \varepsilon > 0$ such that on $(t_0 - \varepsilon, t_0 + \varepsilon)$, $Y \in \Omega^0(\gamma^*TM)$ is the restriction of a vector field X on M, then:

$$\nabla_{\frac{\partial}{\partial t}} Y\Big|_{t_0} = \nabla^{\mathit{aff}}_{\gamma'(t_0)}(X)\Big|_{t_0}$$

where as usual, ∇^{aff}_X is the covariant derivative along X of this connexion.

Remark: We would want to just take (ii) as the definition of $\nabla_{\frac{\partial}{\partial t}}$, but no such X need to exist (e.g. if Y had dense image). [γ on a compact interval would do the job, though.]

Proof. Note that any $Y(t) \equiv Y_t \in (\gamma^*TM)_t := T_{\gamma(t)}M$ takes the form:

$$Y(t) = \sum_i Y_i \frac{\partial}{\partial x_i}\Big|_{\gamma(t)}$$

by definition of the pullback. This shows that any Y is a linear combination of the vector fields $\left\{ \frac{\partial}{\partial x_i}\Big|_{\gamma(t)} \right\}$, which do extend to all of M, **provided** we work locally with respect to some local coordinates $x_1, \ldots, x_n$ on M.

So **<u>if</u>** $\nabla_{\frac{\partial}{\partial t}}$ exists and has the correct properties, then let $\gamma(t) = (x_1(t), \ldots, x_n(t))$, and so:

$$\gamma'(t) = \sum_j x_j'(t) \cdot \frac{\partial}{\partial x_j}\Big|_{\gamma(t)}.$$

Then the defining properties (i) and (ii) say:

$$\nabla_{\frac{\partial}{\partial t}} Y\Big|_t = \nabla_{\frac{\partial}{\partial t}}\left(\sum_j \big(Y_j(t)\big) \cdot \frac{\partial}{\partial x_j}\Big|_{\gamma(t)} \right) = \sum_j Y_j'(t) \frac{\partial}{\partial x_j} + \sum_j Y_j(t) \cdot \nabla_{\frac{\partial}{\partial t}}\left(\frac{\partial}{\partial x_j} \right)$$

by property (i). So as $\frac{\partial}{\partial x_i}$ is restricted from a vector field on TM, by property (ii) we have:

$$= \sum_i \left(Y_i'(t) + \sum_{j,k} \Gamma^i_{jk} x_k'(t) Y_j(t) \right) \frac{\partial}{\partial x_i},$$

where $(\Gamma^i_{jk})_{i,j,k}$ are the Christoffel symbols of ∇^{aff}(vii). Here, we have used:

$$\nabla_{\frac{\partial}{\partial t}} \left(\frac{\partial}{\partial x_j} \right) = \nabla^{\text{aff}}_{\gamma'(t)} \left(\frac{\partial}{\partial x_j} \right) = \sum_k x_k'(t) \nabla^{\text{aff}}_{\partial_k}(\partial_j),$$

where we have used the local expression for $\gamma'(t)$(viii). Hence, as $\{\partial_i\}_i$ are a local basis here, by definition of the covariant derivative and connexion matrix/Christoffel symbols, we have

$$\nabla^{\text{aff}}_{\partial_k}(\partial_j) = \nabla^{\text{aff}}(\partial_k, \partial_j) = \sum_j \Gamma^i_{jk} \frac{\partial}{\partial x_i}$$

[**Exercise** to check]. So hence if $\nabla_{\frac{\partial}{\partial t}}$ exists, then it must be given by the above expression. Then by taking this as our definition, one can check that this does indeed satisfy the conditions [**Exercise** to check]. Then done.

$\square$

Definition 5.1. *If M has an affine connexion A, then we say that $\gamma : (a, b) \to M$ is a* ***geodesic for A*** *if:*

$$\nabla_{\frac{\partial}{\partial t}} \left(\gamma_* \left(\frac{\partial}{\partial t} \right) \right) = 0,$$

where $\nabla_{\frac{\partial}{\partial t}}$ is the connexion given by Lemma 5.1 applied to this affine connexion A. [Note that $\gamma^*(\partial/\partial t) = \partial\gamma/\partial t = \dot{\gamma}$.]

Exercise: Let $m \in M$ and $v \in T_m M$. Then show that there is a local solution to this geodesic equation, $\gamma : (-\varepsilon, \varepsilon) \to M$, such that $\gamma(0) = m$, $\gamma'(0) = v$,

i.e. **locally we can find a geodesic at any point in any direction** [this is just solving the system of differential equations (one for each component) using the expression above. You should find that here, $Y = \gamma_*(\partial/\partial t) = \sum_j x_j'(t) \cdot \frac{\partial}{\partial x_j}\Big|_{\gamma(t)}$, and so $Y_i(t) = x_i'(t)$. Hence using the proof of Lemma 5.1 we find the geodesic equations are $x_i''(t) + \sum_{j,k} \Gamma^i_{jk} x_j' x_k' = 0$].

The case of most interest is when (M, g) is Riemannian, and A is the LC connexion for g. We restrict to this case now.

(vii) Recall that the Christoffel symbols are defined by: $\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{ij} \partial_k$ (by summation convention, must have 1 upper index and 2 lower indices.

(viii) We know that $\nabla_{fX} = f\nabla_X$ and $\nabla_{X_1+X_2} = \nabla_{X_1} + \nabla_{X_2}$. So:

$$\nabla^{\text{aff}}_{\gamma'(t)} = \nabla^{\text{aff}}_{\sum_j x_j'(t) \cdot \frac{\partial}{\partial x_j}\big|_{\gamma(t)}} = \sum_j x_j'(t) \nabla^{\text{aff}}_{\frac{\partial}{\partial x_j}\big|_{\gamma(t)}}.$$

Notation: Let $\boldsymbol{\Omega} \subset TM$ be the open neighbourhood of the zero-section (written $0_M \subset TM$) of points (m, v) such that the corresponding geodesic $\gamma(0) = m$ and $\gamma'(0) = v$ is defined on an interval $(-\varepsilon, 1+\delta)$, for some $\delta > 0$ (i.e. define for some time interval larger than 1).

Note that if $v = 0$, then the geodesic is just the constant map, and so the geodesic is defined for all time.

Definition 5.2. *We say the Riemannian manifold (M, g) is **geodesically complete** if $\Omega = TM$,*

i.e. if all geodesics exist for all time[ix]*(as we can patch together geodesics).*

Definition 5.3. *The **exponential map** is the map: $\exp : \Omega \to M$ sending $(v, m) \mapsto \gamma_v(1)$, where γ_v is the unique geodesic through m with tangent v at $t = 0$.*

We write $\exp_m$ for: $\exp|_{\Omega \cap T_m M}$, i.e. restricting exp just to the tangent space at m. Hence all such geodesics will start at m.

By standard properties of solutions to ODEs, we have the following properties of exp:

(i) exp is smooth

(ii) By the inverse function theorem, the map $\Phi : \Omega \to M \times M$, sending $\Phi(m, v) = (m, \exp_m(v))$, is a local diffeomorphism from a neighbourhood W of $0_M \in TM$ to a neighbourhood of $\Delta \subset M \times M$ (the diagonal).

Note: $\Phi(m, 0) = (m, m)$, and $D(\exp_x)|_0 = \text{id}$, and so hence in a local basis we have:

$$D\Phi_{m,0} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

which is invertible (so can use the inverse function theorem).

(iii) For $m \in M$, $\exists$ a neighbourhood $U \ni m$ and $\varepsilon > 0$ such that if $x, y \in U$, $\exists!$ vector $v \in T_x M$ with $g(v, v) \leq \varepsilon$ and $\exp_x(v) = y$.

(i.e. can locally join points by a geodesic. This is immediate from (ii), since Φ is a local diffeomorphism and so is surjective - inverse function theorem).

Remark: If G is a Lie group (so a manifold) and g is a bi-invariant metric on G (with respect to the Lie group), then exp with respect to g is the usual exponential map of the Lie group which we saw before in §1.2 (Corollary 1.2). So these notions agree.

For example, if (G, g) is geodesically complete and $\exp_g : T_e G \to G$ as above agrees with $\exp : \mathfrak{g} \to G$ as in Corollary 1.2, at least for a suitable rescaling of g (recall $\mathfrak{g} = T_e G$ here).

[ix]Compare this with: "more speed for less time", i.e. rescale v. What this means is that rescaling a finite time interval, we get what appears to be a geodesic defined on a larger time interval. But it will still be a finite time interval. So finite time lengths aren't too interesting here.

So think of exp as giving "distinguished" local charts on (M, g), via the $\exp_m$-images of small balls in $T_m M$ (i.e. map a small ball about $0 \in T_m M$ to all the possible geodesic endpoints of geodesics started at m in those directions).

Hence from exp, we acquire two natural local systems of coordinates on (M, g):

(a) **Geodesic Normal Coordinates:**

Take $m \in M$ and $\varepsilon > 0$ such that: $\exp_m : B_\varepsilon(0) \to M$ is a diffeomorphism onto its image (here, $B_\varepsilon(0) \subset (T_m M, g_m)$ is the ball here with respect to the inner product g_m).

So let $e_1, \ldots, e_n$ be an orthonormal basis of $T_m M$. Then under exp, these give $x_1, \ldots, x_n$ coordinates on a neighbourhood of $m \in M$.

Then if $c_i(t) := \exp_m(te_i) = \gamma_{e_i}(t)$, and if $v = \sum_{i=1}^n v_i e_i \in T_m M$, then the geodesic through m in the direction of v is:

$$c_v(t) = (x_1(t), \ldots, x_n(t)) = (tv_1, \ldots, tv_n),$$

i.e. a linear parallel to v (RHS is w.r.t. coordinates in M). Clearly $c_v(0) = 0$ and $c_v'(0) = v$.

But the general equation for geodesics in local coordinates $\{x_1, \ldots, x_n\}$ on any (M, g) is (from before, using the expression for $\nabla_{\frac{\partial}{\partial t}}$ found in the proof of Lemma 5.1):

$$x_i''(t) + \sum_{j,k} \Gamma^i_{jk} x_j'(t) x_k'(t) = 0$$

for each i. So hence in these coordinates we know $x_i(t) = tv_i$, and so hence this gives:

$$\sum_{j,k} \Gamma^i_{jk} v_j v_k = 0.$$

So evaluating at $t = 0$, so at $m \in M$, as these form a basis we get:

$$\Gamma^i_{jk}\Big|_m = 0,$$

i.e. $\exists$ coordinates on (M, g) such that the Christoffel symbols of A_{LC} vanish at a chosen point m. These coordinates are called **geodesic normal coordinates**.

So in the geodesic normal coordinates, we take Cartesian coordinates and used exp to generate a nice coordinate system on M locally. Another flavour of local coordinates to take are polar coordinates on $\mathbb{R}^n \cong T_m M$ rather than Cartesian ones. This gives rise to our next nice coordinates system for M:

(b) **Geodesic Polar Coordinates:**

Here the coordinates are: $\{r, \theta_1, \ldots, \theta_{n-1}\}$, where the θ_i are the coordinates on $S^{n-1} \subset T_m M$ and r is the radial coordinates. So we have a map

$$\underbrace{S^{n-1}}_{\ni \{\theta_i\}_i} \times \underbrace{(0, \varepsilon)}_{\ni r} \to M, \quad \text{via} \quad (\omega, r) \mapsto \exp_m(r\omega).$$

We will now spend some time proving properties about the usefulness of geodesic polar coordinates.

Lemma 5.2 (Gauss' Lemma). *In geodesic polar coordinates, the Riemannian metric takes the form:*

$$g = \mathrm{d}r \otimes \mathrm{d}r + \sum_{\alpha,\beta} g_{\alpha\beta}(r,\theta)\mathrm{d}\theta_\alpha \otimes \mathrm{d}\theta_\beta.$$

More explicitly, this says:

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1 \quad \textit{and} \quad g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_\alpha}\right) = 0 \ \ \forall \alpha.$$

Caution: This is **not** saying that the metric is a product metric on $\exp\left(S^{n-1} \otimes (0,\varepsilon)\right)$, as the coefficients $g_{\alpha\beta}$ can depend on r.

Proof. The integral curves of $\frac{\partial}{\partial r}$ in our neighbourhood $U = \exp_m(B_\varepsilon(0)) \ni m$ of m are the images of radial straight lines in $B_\varepsilon(0)$ under exp, and so are geodesics (as such lines are geodesics in $B_\varepsilon(0)$).

Now for any geodesic γ, note that, by the compatibility of the LC connexion with the metric, and by property (ii) of $\nabla_{\frac{\partial}{\partial t}}$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\dot\gamma, \dot\gamma) = g\left(\nabla_{\frac{\partial}{\partial t}}(\dot\gamma), \dot\gamma\right) + g\left(\dot\gamma, \nabla_{\frac{\partial}{\partial t}}(\dot\gamma)\right) = 0$$

where both terms on the RHS are zero since $\nabla_{\frac{\partial}{\partial t}}(\dot\gamma) = 0$ by the geodesic equation. So hence geodesics have constant length/speed. So hence this shows that $g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$ is constant, and is equal to 1 at the origin, i.e. on T_mM. So hence we have $g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1$.

For the second condition, let us fix $0 \in U$ and consider $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta_\alpha}$ (for some fixed α) along the line from m to p (as seen in $\exp_m^{-1}(U) = B_\varepsilon(0)$). Then, again by compatibility of the LC connexion (choosing the vector fields to be $X = \frac{\partial}{\partial r}$, etc)

$$\frac{\partial}{\partial r}\left(g\left(\frac{\partial}{\partial \theta_\alpha}, \frac{\partial}{\partial r}\right)\right) = g\left(\nabla_{\frac{\partial}{\partial r}}\left(\frac{\partial}{\partial \theta_\alpha}\right), \frac{\partial}{\partial r}\right) + \underbrace{g\left(\frac{\partial}{\partial \theta_\alpha}, \nabla_{\frac{\partial}{\partial r}}\left(\frac{\partial}{\partial r}\right)\right)}_{=0 \text{ by geodesic equation for radial line, i.e. } \nabla_{\frac{\partial}{\partial r}}\left(\frac{\partial}{\partial r}\right)=0}$$

$$= g\left(\nabla_{\frac{\partial}{\partial \theta_\alpha}}\left(\frac{\partial}{\partial r}\right), \frac{\partial}{\partial r}\right),$$

where again using the fact that since the LC connexion is torsion-free, we have:

$$\nabla_{\frac{\partial}{\partial r}}\left(\frac{\partial}{\partial \theta_\alpha}\right) - \nabla_{\frac{\partial}{\partial \theta_\alpha}}\left(\frac{\partial}{\partial r}\right) - \underbrace{\left[\frac{\partial}{\partial \theta_\alpha}, \frac{\partial}{\partial r}\right]}_{=0} = \tau\left(\frac{\partial}{\partial \theta_\alpha}, \frac{\partial}{\partial r}\right) = 0.$$

So hence we have:

$$\frac{\partial}{\partial r}\left(g\left(\frac{\partial}{\partial \theta_\alpha}, \frac{\partial}{\partial r}\right)\right) = \frac{1}{2} \cdot \frac{\partial}{\partial \theta_\alpha}\underbrace{\left(g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\right)}_{=1 \text{ by the above}} = 0.$$

Hence $g\left(\frac{\partial}{\partial \theta_\alpha}, \frac{\partial}{\partial r}\right)$ is constant with r. But as $r \to 0$, this $\to 0$, since on T_mM (i.e. where $r = 0$), $\frac{\partial}{\partial \theta_\alpha}$ and $\frac{\partial}{\partial r}$ are orthogonal. So hence done. □

Remarks:

(i) We can define for $\gamma : (a,b) \to M$ the **length of** γ by the integral of the norm of its tangent vector, i.e.

$$L(\gamma) := \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_g^{1/2}\, \mathrm{d}t$$

where $\langle \cdot,\cdot\rangle_g = g(\cdot,\cdot)$. Then we can define a **metric** $\mathrm{d}(p,q)$ for $p,q \in M$ to be the infimum of all lengths of curves in M from p to q. This yields a metric, which is compatible with/induces the same topology on (M,g) as the one initially placed on M. [See Example Sheet 3 for details.]

So in geodesic polars, if $p,q \in U$ (U as above/before), if p is the centre of this chart, then we can show that the radial path from p to q is length-minimising (i.e. map the radial path in $B_\varepsilon(0)$ into M via exp - this path is length-minimising). Indeed, if $\tilde{\gamma}$ is any path from p to q in M, then:

- If $\tilde{\gamma} \subset U$, then we can use geodesic polar coordinates about p to describe the entirety of $\tilde{\gamma}$, and so by Lemma 5.2:
$$L(\tilde{\gamma}) = \int_{\tilde{\gamma}} \sqrt{(\dot{r})^2 + \langle \dot{\theta}_\alpha, \dot{\theta}_\beta\rangle}\, \mathrm{d}t \geq \int_{\tilde{\gamma}} |\dot{r}|\, \mathrm{d}t = r(q),$$
where $r(q)$ is the radial length from the centre (p) to q. Moreover, we have equality only if the $\dot{\theta}_{\tilde{\gamma}(t)}$-terms vanish, and if $\dot{r} > 0$, i.e. if $\tilde{\gamma}$ is a radial line moving outwards.
- If $\tilde{\gamma} \not\subset U$, then $\tilde{\gamma}$ exits through $\exp_p\left(\partial B_{r(q)}(0)\right)$ at some point, and so hence $L(\tilde{\gamma}) \geq r(q)$, by the triangle inequality for the metric here.

So hence this proves this claim.

(ii) If we consider a set U as above, and a geodesic γ from p to q which is **not** radial, then if $\varepsilon > 0$ was sufficiently small (i.e. the time-length of γ), the maximum value of r along the geodesic γ is achieved at an **end point**.

Why is this? Well, consider the map $\varphi : (-\varepsilon,\varepsilon) \times STU \to M$ with respect to g (here, STU is the sphere tangent bundle) such that the map: $\varphi(\cdot,(q,v))$ is the geodesic through (q,v), with $|v| = 1$, parameterised at unit speed.

Then let us consider: $F(t,q,v) := \left|\exp_m^{-1}(\varphi(t,q,v))\right|^2 = r^2$ (i.e. $F = g(r,r)$). Then, $\frac{\partial F}{\partial t} = 2g\left(\frac{\partial r}{\partial t}, r\right)$, and so

$$\frac{\partial^2}{\partial t^2}F = 2g\left(\frac{\partial^2 r}{\partial t^2}, r\right) + 2\left|\frac{\partial r}{\partial t}\right|^2,$$

since we know that the connexion is compatible with g. But then it we set $q = m =$ centre of the chart, then we have $r(t,m,v) = tv$, and so:

$$\left.\frac{\partial^2 F}{\partial t^2}\right|_{0,m,v} = 2|v|^2 = 2 > 0.$$

So hence by continuity, $\exists$ a neighbourhood Ω of m in U such that $\left.\frac{\partial^2 F}{\partial t^2}\right|_{0,q,v} > 0$ for all $q \in \Omega$, and so hence this stays positive in $(-\tilde{\varepsilon},\tilde{\varepsilon}) \times ST\Omega$, if $\tilde{\varepsilon} > 0$ is sufficiently small.

So as the radial coordinate is convex (as it lies in a convex set), if it had an interior local maximum, we would get a point such that $\frac{\partial r}{\partial t} = 0$ and $\frac{\partial^2 r}{\partial t^2} \leq 0$, i.e. $\frac{\partial^2 F}{\partial t^2} \leq 0$, a contradiction. $\square$

Note: This shows that, for $\varepsilon > 0$ sufficiently small, $\exp_m(B_\varepsilon(0))$ is **geometrically convex**, i.e. $\forall p, q \in \exp_m(B_\varepsilon(0))$, $\exists$ a **length-minimising** curve between p, q which lies entirely in the ball (as opposed to the usual notion of convexity, which would be a straight line joining them lying in this set).

[Compare this with the idea of a manifold being of "finite type", as in §2.4.]
i.e. on a compact manifold, as these neighbourhoods cover and are compatible on overlaps, we get a finite subcover with these properties, etc.

These observations leave to:

Proposition 5.1 (Hopf-Rinow)**.** *If (M, g) is geodesically complete, then $\forall p, q \in M$, $\exists$ a length-minimising geodesic from p to q.*

Remark: It is a fact that:

$$M \text{ is geodesically complete} \iff (M, d) \text{ is complete as a metric space}$$

where d was the metric defined before, via the minimal length of curves.

In particular, this tells us that if M is any compact manifold, then we get strong existence theorems for geodesics in M (as we know it has some Riemannian structure, and then (M, d) is always complete, and so it is geodesically complete, and so hence by Hopf-Rinow we get global existence of length-minimising geodesics).

Proof. For $m \in M$ and $\exp_m : T_m M \to M$ (defined on all of $T_m M$ as M is geodesically complete), we will show that each $q \in M$ can be joined to m by a minimal geodesic.

By Remark (ii) above, we know that $\exists$ a geodesically convex neighbourhood U of m, such that each $p, q \in U$ can be joined by a geodesic of length $d(p, q)$.

Claim: If $p, q \in M$ and δ is sufficiently small, then $\exists p_0 \in \partial B_\delta^{(g)}(p) = \{\tilde{p} : d(p, \tilde{p}) = \delta\}$ such that:

$$d(p, p_0) + d(p_0, q) = d(p, q).$$

Proof of Claim. If $\delta > 0$ is sufficiently small, then we know (as exp is a diffeomorphism here) $\partial B_\delta^{(g)}(p) = \exp_p(\partial B_\delta(0))$, where $B_\delta(0)$ is an open ball in $\mathbb{R}^n = T_m M$.

So hence by compactness of this sphere, $\exists p_0 \in B_\delta^{(g)}(p)$ such that $d(q, B_\delta^{(g)}(p)) = d(q, p_0)$, as the LHS is defined as an infimum of distances over the sphere, and so we can find a sequence tending to this infimum, and so can extract a convergent subsequence by compactness, etc.

So hence if γ is any curve from p to q, we have:

$$\begin{aligned} L(\gamma) &\geq d(p,\gamma(t)) + d(\gamma(t),q) \\ &\geq d(p,\gamma(t)) + d(p_0,q) \\ &= d(p,p_0) + d(p_0,q). \end{aligned}$$

Indeed, the first line is from the triangle inequality, where $\gamma(t)$ is the last intersection point of γ with this sphere (which must exist for γ to reach q). The second line comes from the definition of p_0 being the infimum, noting that $\gamma(t) \in B_\delta^{(g)}(p)$. Then the final line comes from the fact that we have by definition of $B_\delta^{(g)}(p)$, $d(p,p_0) = \delta = d(p,\gamma(t))$.

So hence taking the infimum over all γ gives:

$$d(p,q) \geq d(p,p_0) + d(p_0,q).$$

But then the lines $p \rightsquigarrow p_0 \rightsquigarrow q$ is clearly a path $p \rightsquigarrow q$, so the other inequality follows from the triangle inequality for d. So hence we have:

$$d(p,q) = d(p,p_0) + d(p_0,q),$$

as required.

$\square$

[The idea of the proof of this claim is that this minimum distance is attained on spheres, and on small enough spheres we have geodesic polar coordinates.]

So now take $p = m \in M$ such that $\exp_m$ is defined on all of T_mM, and set $p_0 = m_0$, and pick $v \in T_mM$ such that $\exp_m(\delta \cdot v) = m_0$. Then consider $\gamma(t) = \exp_m(tv)$, which is defined for all time (i.e. local extension, which exists for all time due to geodesically complete, gives us a goedesic. We will show this takes us to q in time $d(m,q)$ which implies minimality).

Let $I = \{t \in \mathbb{R} : d(q,\gamma(t)) + t = d(q,m)\}$. So as by the above, $d(m,m_0) + d(m_0,q) = d(m,q)$, we know that $\delta \in I$.

Then if $T = \sup(I \cap [0,d(m,q)])$, we want to show that $T = d(m,q)$. Note that as I is closed, we have $T \in I$ (as $T \in I \cap [0,d(m,q)]$ as this is closed).

So suppose $T < d(m,q)$, and consider our above claim at $\gamma(T)$. Then, $\exists \varepsilon > 0$ and $p_0 \in M$ such that $d(\gamma(T),p_0) = \varepsilon$ and:

$$d(p_0,q) = d(\gamma(T),q) - d(\gamma(T),p_0) = \underbrace{(d(q,m) - T)}_{=d(\gamma(T),q) \text{ as } T\in I} - \varepsilon = d(m,q) - T - \varepsilon.$$

Therefore, we have by the triangle inequality,

$$d(m,p_0) \geq d(m,q) - d(q,p_0) = T + \varepsilon.$$

So let $\lambda(t)$ be the minimal geodesic from $\gamma(T)$ to p_0 (exists by the claim). So hence:

$$\text{length}\left(\gamma|_{[0,T]}\right) + \text{length}(\lambda) = T + \varepsilon \overset{\text{by above}}{\leq} d(m,p_0)$$

But then as this is a path $m \rightsquigarrow p_0$ and d is the inf of all such paths, the inequality here must be an equality. Hence this $\Rightarrow$ λ extends $\gamma|_{[0,T]}$. Then, $p_0 = \gamma(T+\varepsilon)$, and so $T+\varepsilon \in I$, a contradiction to T being the supremum.

Hence we must have $T = d(m,q)$, and so $d(m,q) \in I$, and so hence q lies on a minimal geodesic from m. So done.

□

Definition 5.4. *The **energy** of a path $\gamma : (a,b) \to M$ is:*

$$E(\gamma) = \int_a^b |\dot{\gamma}(t)|_g^2 \, \mathrm{d}t$$

where $|\cdot|_g^2 := g(\cdot,\cdot)$ is the norm with respect to g.

Remarks:

(i) Unlike length, the energy depends on the parameterisation (i.e. how 'fast' one travels along a path. This is seen by considering the energy of $\tilde{\gamma}(t) = \gamma(f(t))$ for some $f : (a,b) \to (a,b)$ smooth).

(ii) A Cauchy-Schwarz inequality argument shows that energy minimising γ are geodesics ($\equiv$ critical points of the length functional).

Let $c_0 = c_0(t)$ be a curve in M, and consider $c(t,s) = c_s(t)$, where $\{c_s(t)\}_{s\in(-\varepsilon,\varepsilon)}$ is a variation of c_0. Let $\dot{c} = \frac{\partial c}{\partial t}$ and $c' = \frac{\partial c}{\partial s}$. Then our aim is to compute $\frac{\mathrm{d}^2}{\mathrm{d}s^2} E(c_s)\Big|_{s=0}$, assuming that c_0 is a geodesic.

So as:

$$E(c_s) = \int_a^b g\left(\dot{c}|_{t,s}, \dot{c}|_{t,s}\right) \, \mathrm{d}t,$$

we have, since the LC connexion is compatible with g, and by definition of $\nabla_{\frac{\partial}{\partial s}}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} E(c_s) &= \int_a^b \frac{\partial}{\partial s} g(\dot{c},\dot{c}) \, \mathrm{d}t \\
&= 2\int_a^b g\left(\nabla_{\frac{\partial}{\partial s}}(\dot{c}), \dot{c}\right) \, \mathrm{d}t \\
&= 2\int_a^b g\left(\nabla_{\frac{\partial}{\partial t}}(c'), \dot{c}\right) \\
&= 2\int_a^b \left[\frac{\partial}{\partial t}\left(g(c',\dot{c})\right) - g(c', \nabla_{\frac{\partial}{\partial t}}(\dot{c}))\right] \mathrm{d}t \\
&= 2g(c',\dot{c})|_{t=a}^{t=b} - 2\int_a^b g(c', \nabla_{\frac{\partial}{\partial t}}(\dot{c})) \, \mathrm{d}t,
\end{aligned} \tag{5.1}$$

where in (5.1), we used again that the LC connexion is torsion-free, which gives:

$$\nabla_{\frac{\partial}{\partial s}}(\dot{c}) = \nabla_{\frac{\partial}{\partial t}}(c'), \tag{5.2}$$

and in the line after we again used the compatibility with g. In the last line, we have simply integrated the first term. This expression is called the **first variation formula**.

To calculate the second variation, note that from (5.1), we have:

$$\begin{aligned}\frac{1}{2}\cdot\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(c_s) &= \int_a^b \frac{\partial}{\partial s} g\left(\nabla_{\frac{\partial}{\partial t}}(c'), \dot{c}\right)\,\mathrm{d}t\\ &= \int_a^b g\left(\nabla_{\frac{\partial}{\partial t}}(c'), \nabla_{\frac{\partial}{\partial t}}(c')\right)\,\mathrm{d}t + \int_a^b g\left(\nabla_{\frac{\partial}{\partial s}}\left(\nabla_{\frac{\partial}{\partial t}} c'\right), \dot{c}\right)\,\mathrm{d}t \quad \text{using (4)},\\ &= \int_a^b g\left(\nabla_{\frac{\partial}{\partial t}}(c'), \nabla_{\frac{\partial}{\partial t}}(c')\right)\,\mathrm{d}t + \int_a^b g\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}(c'), \dot{c}\right)\,\mathrm{d}t - \int_a^b g\left((R(\dot{c}, c'))(c'), \dot{c}\right)\,\mathrm{d}t,\end{aligned}$$

where in the last line, we have used the fact that $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, and that $[X,Y] = 0$ for coordinate vector fields, i.e. $X = \frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial s}$ (and using (ii) from Lemma 5.1).

So hence if c_0 is a geodesic, then $\nabla_{\frac{\partial}{\partial t}}(\dot{c})\Big|_{(t,0)} = 0$, and so this becomes:

$$\begin{aligned}\frac{1}{2}\cdot\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(c_s)\Big|_{s=0} = \int_a^b g\left(\nabla_{\frac{\partial}{\partial t}}(c')\Big|_{(t,0)}, \nabla_{\frac{\partial}{\partial t}}(c')\Big|_{(t,0)}\right)\,\mathrm{d}t - \int_a^b g\left((R(\dot{c}, c'))(c'), \dot{c}\right)\Big|_{s=0}\,\mathrm{d}t\\ + g\left(\nabla_{\frac{\partial}{\partial s}}(c'), \dot{c}\right)\Big|_{(a,0)}^{(b,0)},\end{aligned}$$

where we have been able to integrate the second term in the above, since

$$\frac{\partial}{\partial t} g\left(\nabla_{\frac{\partial}{\partial s}}(c'), \dot{c}\right) = g\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}(c'), \dot{c}\right) + \underbrace{g\left(\nabla_{\frac{\partial}{\partial s}}, \nabla_{\frac{\partial}{\partial t}}(\dot{c})\right)}_{=0 \text{ as } \nabla_{\frac{\partial}{\partial t}}(\dot{c})=0},$$

and so we can integrate the LHS of this as it is a total time derivative.

This expression is the **second variation formula**.

Theorem 5.1 (Myers' Theorem). *Let (M,g) be a connected Riemannian manifold which is complete (i.e. (M,d) is a complete metric space). Suppose that we have a curvature bound of the form:*

$$Ricci \geq \frac{(n-1)g}{r^2}$$

for some $r > 0$. Then, we have $diam(M,g) := \sup_{p,q\in M} d(p,q) \leq \underbrace{\pi r}_{=diam(S^n(r))} < \infty$.

In particular, the universal cover of M has finite diameter, and so hence $|\pi_1(M)| < \infty$, and so hence M is automatically compact.

Note: The condition here is that the Ricci curvature is uniformly bounded on unit length vectors in g, by $(n-1)/r^2$, where n is the dimension of M. So hence if the Ricci curvature has any such uniform bound, then we can always find an r such that the above is true. So hence the assumption is just that the Ricci curvature is uniformly bounded.

Remark: A sphere of radius r has constant Ricci curvature $(n-1)/r^2$. So hence Myers' theorem is just saying that if (M,g) 'curves' more than a sphere of radius r, then $\text{diam}(M) \leq \text{diam}(S^n_r)$ (i.e. its diameter is bounded by that of a sphere of radius r).

Proof. We know that M is complete, and so hence (from before) M is geodesically complete. So now fix $L < \text{diam}(M,g)$ (note we currently do not know $\text{diam}(M,g) < \infty$). So hence, as the diameter is a supremum, $\exists\ p,q \in M$ such that $d(p,q) = L$. Then by Hopf-Rinow, $\exists$ a length-minimising geodesic γ between p and q.

Now if Y is a vector field along γ which vanishes at the endpoints p,q, then $\exists$ a variation γ inducing Y[(x)]. Thus for this variation $\{c_s^Y\}_{s\in(-\varepsilon,\varepsilon)}$, which we denote the curve by c instead of γ (to match what we did before), we can study the second variation:

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(c_s^Y)\bigg|_{s=0} =: I(Y,Y),$$

and we know that $I(Y,Y) \geq 0$, as $c_0 = \gamma$ is a geodesic (i.e. geodesics are minima of this energy functional, and so the second variation must be ≥ 0).

So let $e_1,\dots,e_n$ be an orthonormal basis of T_pM. Wlog we can choose $e_1 := \dot\gamma(0)$ $(\neq 0)$. Then via parallel transport along γ, we get an orthonormal basis:

$$t \mapsto \{X_1(t) = \dot\gamma(t), X_2(t), \dots, X_n(t)\}$$

of vector fields along γ (i.e. at t this gives a basis of the tangent space $T_{\gamma(t)}M$). Note we know that $X_i(0) = e_i$. [We assume that $\gamma : [0,L] \to M$ is parameterised by arc-length.

So define/choose the vector field along γ via:

$$Y_i(t) := \sin\left(\frac{\pi t}{L}\right)X_i(t).$$

Then these vanish at p (i.e. when $t=0$) and at q (when $t=L$). Now using our second variation formula before, we get:

$$I(Y_i,Y_i) = -\int_0^L g(Y_i,\ddot Y_i)\ \mathrm{d}t - \int_0^L g\left(R(\dot\gamma,Y_i)Y_i,\dot\gamma\right)\ \mathrm{d}t,$$

where we have used:

$$g(\dot Y,\dot Y) = \frac{\mathrm{d}}{\mathrm{d}t}\big(g(Y,\dot Y)\big) - g(Y,\ddot Y)$$

for the first term (which comes from compatibility with the metric)[(xi)]. Then recall that we also have:

$$g\left(R(X,Y)Z,W\right) = R(W,Z,X,Y).$$

[(x)] i.e. $\exists$ a variation $\{\gamma_s(t)\}_{s\in(-\varepsilon,\varepsilon)}$ of $\gamma=\gamma_0$ such that $Y|_t = \frac{\partial}{\partial s}\gamma_s(t)\big|_{(t,0)}$, i.e. for a variation of γ, for fixed t, consider $\gamma_s(t)$. Then, $\frac{\partial}{\partial s}\gamma_s(t)\big|_{s=0}$ gives a tangent vector at γ, so defines a vector field. But as for a variation the endpoints are fixed, the tangent vector must be zero at them.

[(xi)] When we write $\dot Y_i$ here, we mean $\nabla_{\partial/\partial t}Y_i$. So as the X_i are covariant constant, by Lemma 5.1 (ii) we have $\nabla_{\partial/\partial t}X_i = 0$ and so hence:

$$\ddot Y_i = \nabla_{\partial/\partial t}\left(\nabla_{\partial/\partial t}Y_i\right) = \cdots = X_i(t)\cdot\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin(\pi t/L)$$

using the Leibniz property of $\nabla_{\partial/\partial t}$.

So hence, inputting our choice of Y gives:

$$I(Y_i, Y_i) = \int_0^L \sin^2\left(\frac{\pi t}{L}\right)\left(\frac{\pi^2}{L^2} - R(\dot{\gamma}, X_i, \dot{\gamma}, X_i)\right) \, \mathrm{d}t$$

where we have used the symmetries/properties of R, etc. Now sum over i to get:

$$\sum_i I(Y_i, Y_i) = \int_0^L \sin^2\left(\frac{\pi t}{L}\right)\left((n-1)\cdot\frac{\pi^2}{L^2} - \mathrm{Ricci}(\dot{\gamma}, \dot{\gamma})\right) \, \mathrm{d}t,$$

where we have used the definition of the Ricci curvature as a trace. So then note that we know $I(Y_i, Y_i) \geq 0$ for each i, and so hence:

$$\int_0^L \sin^2\left(\frac{\pi t}{L}\right)\left((n-1)\cdot\frac{\pi^2}{L^2} - \mathrm{Ricci}(\dot{\gamma}, \dot{\gamma})\right) \, \mathrm{d}t \geq 0.$$

By assumption, we know:

$$\mathrm{Ricci}(\dot{\gamma}, \dot{\gamma}) \geq \frac{(n-1)}{r^2} g(\dot{\gamma}, \dot{\gamma}) = \frac{n-1}{r^2},$$

since γ is arc-length parameterised and so $g(\dot{\gamma}, \dot{\gamma}) = 1$ (as g is the inner product). So hence this shows:

$$\int_0^L \sin^2\left(\frac{\pi t}{L}\right)\left((n-1)\cdot\frac{\pi^2}{L^2} - \frac{n-1}{r^2}\right) \, \mathrm{d}t \geq \int_0^L \sin^2\left(\frac{\pi t}{L}\right)\left((n-1)\cdot\frac{\pi^2}{L^2} - \mathrm{Ricci}(\dot{\gamma}, \dot{\gamma})\right) \, \mathrm{d}t \geq 0,$$

and so hence

$$(n-1)\cdot\frac{\pi^2}{L^2} - \frac{n-1}{r^2} \geq 0 \quad \Rightarrow \quad L \leq \pi r.$$

So hence we see: $L < \mathrm{diam}(M, g) \Rightarrow L \leq \pi r$. So hence as the diameter is just the supremum over all such L, we see that: $\mathrm{diam}(M, g) \leq \pi r$. So done.

$\square$

6. The Yang-Mills Equation

This section is non-examinable

Note that with differential forms, we:

- Study the algebra $\Omega^\star(M)$ via $\mathrm{H}^\star_{\mathrm{dR}}(M)$, which is a topological invariant.
- Study specific forms, e.g. symplectic forms, volume forms, with their own geometry.

With connexions,

- We mostly study d_A for fixed A (e.g. holonomy), or d_{LC} on (M,g).
- We can also study the information held in the space of connexions, e.g via the **Yang-Mills functional**.

So let $E \to M$ be a bundle, and fix a metric on E (as defined beforre). Then recall that the connexion A on E is **compatible** with g if:

$$\mathrm{d}(g(s,t)) = g(\mathrm{d}_A(s),t) + g(s,\mathrm{d}_A(t)) \quad \forall s,t \in \Gamma(E).$$

Recall then that the connexion matrices θ_α for the connexion are skew-symmetric. From the expression:

$$(F_A)_\alpha = \mathrm{d}\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = \mathrm{d}\theta_\alpha \frac{1}{2}[\theta_\alpha, \theta_\alpha]$$

for the curvature, we also see that F_A is skew-symmetric, i.e. $F_A \in \Omega^2\,(\mathrm{Skew}(\mathrm{End}(E))$. [Compare this with inside $\mathrm{Mat}_n(\mathbb{R})$, where skew-endomorphisms are just the Lie group of $O(n)$, and that this is a Lie subalgebra.]

A metric on E then induces a metric on $\mathrm{End}(E)$ by:

$$|S|^2 = \mathrm{tr}(SS^*) = \sum_{i,j} |S_{ij}|^2,$$

i.e. just the norm-squared of all elements, and so this acts as $-\mathrm{tr}(S^2)$ on $\mathrm{SkewEnd}(E)$ (as here, $S^* = -S$). Similar, a metric/inner product on a vector space V yields metrics on $\Lambda^k V$ for each k.

Then if $\{e_1,\dots,e_n\}$ is an orthonormal basis for V, then we know that $\{e_I = e_{i_1}\wedge\cdots\wedge e_{i_k} : i_1 < \cdots < i_k\}$ forms a basis for $\Lambda^k V$, and we declare that this basis is an orthonormal basis for $\Lambda^k V$ (i.e. the inner product is defined via: $\langle e_I, e_J\rangle = \delta_{IJ}$).

So a metric on M, say g, induces metrics on all differential forms, i.e. on all $\Lambda^k T^*M$.

Definition 6.1. *If ω is a nowhere-zero n-form on M^n, we can then define (given a metric g) the **volume form**:*

$$vol_g := \frac{\omega}{\sqrt{g(\omega,\omega)}},$$

the unit-length nowhere-zero n-form.

Then we get an (L^2-) inner product on $\Omega^k(M)$ via:

$$\langle \alpha, \beta\rangle_{L^2} := \int_M g(\alpha(x), \beta(x))\ \mathrm{vol}_g \qquad \in \mathbb{R},$$

called the **L^2 inner product**, with corresponding norm $\|\alpha\|_{L^2} := \sqrt{\langle \alpha, \alpha\rangle_{L^2}}$.

Given (M, g) and a bundle $E \to M$ with a metric on E, we know that we get a metric on $\Omega^2(\mathrm{End}(E))$, as given before. So hence if $\eta \in \Omega^2(M)$ and $A \in \mathrm{End}(E)$, then we define:

$$|\eta \otimes A|^2 := g(\eta, \eta)|A|^2,$$

and this is a metric on $\Omega^2(M) \otimes \mathrm{End}(E)$.

Definition 6.2. *The **Yang-Mills functional**, $yM_E : \mathbb{A}_E \to \mathbb{R}$ (where $\mathbb{A}_E$ is space of connexions on E), is defined via:*

$$A \longmapsto \|F_A\|^2_{L^2}.$$

6.1. **The Euler-Lagrange Equations (Formally).**

We want to find connexions A that minimise yM_E, i.e. (as the space of connexions is an affine space)

$$0 = \frac{\mathrm{d}}{\mathrm{d}}\bigg|_{t=0} \|F_{A+ta}\|^2_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\|F_A + t\mathrm{d}_{A^*\otimes A}(a) + t^2 a \wedge a\|^2_{L^2}\right)$$
$$= 2\int_M g(F_A, \mathrm{d}_{A^*\otimes A}(a))\ \mathrm{vol}_g,$$

for $a \in \Omega^1(\mathrm{End}(E))$, from our knowledge of F_{A+a}.

> **Fact:** [Analysis Input] The operator $\mathrm{d}_{A^*\otimes A} : \Omega^1(\mathrm{End}(E)) \to \Omega^2(\mathrm{End}(E))$ has a **formal adjoint**, $(\mathrm{d}_{A^*\otimes A})^*$, i.e. it satisfies:
> $$g\left(\mathrm{d}_{A^*\otimes A}(\alpha), \beta\right) = g\left(\alpha, (\mathrm{d}_{A^*\otimes A})^*(\beta)\right)$$
> for all $\alpha \in \Omega^1(\mathrm{End}(E))$, $\beta \in \Omega^2(\mathrm{End}(E))$.

So hence we have:

$$0 = 2\int_M g\left((\mathrm{d}_{A^*\otimes A})^*(F_A), a\right)\ \mathrm{vol}_g$$

for all a, and so this just tells us that:

$$(\mathrm{d}_{A^*\otimes A})^* F_A = 0.$$

This is the **Euler-Lagrange equation for A**.

[Compare this with the 2nd Bianchi identity, which says: $\mathrm{d}_{A^*\otimes A}F_A = 0$ always! So this very similar identity for the Euler-Lagrange equations is a lot more trivial.]

Remark: The exterior derivative $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ also has a formal adjoint, d^* (via the Hodge-star operator). The equations: $\mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$ turn out to be the Euler-Lagrange equations for the functional:

$$\{\alpha \in \Omega^k(M) : \alpha \text{ is closed, and } [\alpha] \in \mathrm{H}^k_{\mathrm{dR}}(M) \text{ is a fixed class}\} \to \mathbb{R} \quad \text{sending} \quad \alpha \longmapsto \|\alpha\|^2_{L^2}.$$

The solutions to the Euler-Lagrange equations here are called **harmonic forms**. These exist and are unique for compact M, giving an analytic approach to thinking about $\mathrm{H}^k_{\mathrm{dR}}(M)$, as the vector space of harmonic forms (i.e. $\mathrm{H}^k_{\mathrm{dR}}(M)$ is the space of extremisers of the above functional).

We will spend the rest of the course considering the case $\dim(M) = 4$. Then, $\exists$ an operator $\star : \Lambda^2 V \to \Lambda^2 V$, called the **Hodge-star operator**, in which on an orthonormal basis for V, is defined via:

$$\star(e_{\sigma(1)} \wedge e_{\sigma(2)}) = \mathrm{sign}(\sigma) \cdot e_{\sigma(3)} \wedge e_{\sigma(4)},$$

for $\sigma \in S_4$, the symmetric group on 4 elements. Hence we see: $\star \circ \star = \mathrm{id}_{\Lambda^2 V}$, which implies $\star$ has eigenvalues of only ± 1.

So hence if (M, g) is a 4-manifold, then we can decompose:

$$\Omega^2(M) = \Omega^2(M)^+ \oplus \Omega^2(M)^-$$

into the ± 1 eigenspaces of $\star$. Hence $\Omega^2(M)^+$ is called the space of **self-dual** 2-forms, and $\Omega^2(M)^-$ the space of **anti-self dual** 2-forms.

Now if A is a connexion on the bundle $E \to M^4$, then:

$$F_A \in \Omega^2(\mathrm{End}(E)) := \Gamma\left(\mathrm{End}(E) \otimes \Lambda^2(T^*M)\right) = \underbrace{\Gamma\left(\mathrm{End}(E) \otimes \Lambda^{2,+}\right)}_{\ni F_A^+} \oplus \underbrace{\Gamma\left(\mathrm{End}(E) \otimes \Lambda^{2,-}\right)}_{\ni F_A^-}$$

from how we know sections interact with direct products. Here, $F_A^\pm$ is the part of the curvature on each space.

Then from the definition of $\star$ (the Hodge-star), it is easy to see/check that [**Exercise**]:

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle \cdot \mathrm{vol}_g$$

(we know the LHS is in $\Omega^4(M) = \mathrm{span}(\mathrm{vol}_g)$, so just need to find constant).

Moreover, in our basis, we know/can check:

$$(\Lambda^2 V)^+ = \mathrm{span}\{e_1 \wedge e_2 + e_3 \wedge e_4,\ e_1 \wedge e_3 + e_2 \wedge e_4,\ e_1 \wedge e_4 \wedge e_2 \wedge e_3\}$$
$$(\Lambda^2 V)^- = \mathrm{span}\{e_1 \wedge e_2 - e_3 \wedge e_4,\ e_1 \wedge e_3 - e_2 \wedge e_4,\ e_1 \wedge e_4 - e_2 \wedge e_3\},$$

and so one can check that, if $\alpha \in (\Lambda^2 V)^+$ and $\beta \in (\Lambda^2 V)^-$, then $\alpha \wedge \beta = 0$ (i.e. these spaces are orthogonal with respect to $\wedge$).

Then if $\alpha \in \Omega^2(M)$, we know we can decompose it into $\alpha = \alpha^+ + \alpha^-$, where $\alpha^\pm \in (\Lambda^2 V)^\pm$ (i.e. $\alpha^+ = \frac{1}{2}(\alpha + \star(\alpha))$, $\alpha^- = \frac{1}{2}(\alpha - \star(\alpha))$). And so:

$$\begin{aligned} \alpha \wedge \alpha = (\alpha^+ + \alpha^-) \wedge (\alpha^+ + \alpha^-) &= \alpha^+ \wedge \alpha^+ + \underbrace{\alpha^+ \wedge \alpha^-}_{=0} + \underbrace{\alpha^- \wedge \alpha^+}_{=0} + \alpha^- \wedge \alpha^- \\ &= \underbrace{\alpha^+ \wedge (\star(\alpha)^*)}_{\text{as } \alpha^+ = \star(\alpha^+)} - \underbrace{\alpha^- \wedge (\star(\alpha^-))}_{\text{as } \alpha^- = -\star(\alpha^-)} \\ &= (|\alpha^+|^2 - |\alpha^-|^2)\mathrm{vol}_g, \end{aligned}$$

from our expression for $\alpha \wedge (\star(\beta))$. But then we know:

$$|\alpha|^2 = |\alpha^+|^2 + |\alpha^-|^2.$$

So hence we see that α is **anti-self dual**, i.e.

$$\alpha^+ = 0 \iff -\alpha^2 = -\alpha \wedge \alpha = |\alpha|^2 \mathrm{vol}_g$$

(i.e. constant multiple of the 'identity' element of this space).

Corollary 6.1. *A connexion A in $E \to M^4$ satisfies:*

$$\|F_A\|_{L^2}^2 =: yM_E(A) \geq \underbrace{\int_M tr(F_A \wedge F_A)}_{\text{independent of } A \text{, by Chern-Weil theory}},$$

with equality $\iff$ A is anti-self dual.

Proof. None given. □

Note: So for example, if the characteristic class (found from Chern-Weil theory, $\mathrm{tr}(F_A^2)$) is negative in $\mathrm{H}^4_{\mathrm{dR}}(M) \cong \mathbb{R}$, then the integral on the RHS is negative, and so we cannot have equality here. Hence E will admit **no** anti-self dual connexion in this case.

In the realm 1983-1986, Donaldson proved the following theorem:

Theorem 6.1 (Donaldson). *Let M be a closed, oriented 4-manifold. Suppose that the cup-product $H^2(M) \times H^2(M) \to \mathbb{Z}$ is positive definite. Then, if $\pi_1(M) = \{1\}$ is trivial, then the cup-product form is diagonalisable, i.e. the identity in some basis.*

Remark: There are non-diagonalisable definite forms, E_8, which are intersection forms of topological 4-manifolds.

Idea of Proof. We introduce a bundle $E \to M$, a metric on M and on E, and we study the space of anti-self dual connexions A on E.

This equation is elliptic, meaning the solution spaces form nice finite-dimensional manifolds (modulo symmetry).

For suitable E (e.g. rank 3), M_{ASD} (ASD = Anti-Self Dual) is a 5-manifold with 2 kinds of boundary:

- Connexions, where $|F_A|$ becomes concentrated at one point of M (these are $\cong M$).
- Connexions whose holonomy ($\subset SO(3)$ as a rank 3 bundle) turns out to lie in $S^1 \subset SO(3)$. This leads to singular ends of M_{ASD} which are cones on $\mathbb{C}P^2$.

Then cobordism $\Rightarrow$ constraints on intersection forms on M. □

End of Lecture Course