Algebraic Topology (Part III)

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These notes are produced entirely from the course I took, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. Please send any corrections to pdtwm2@cam.ac.uk

Recommended books: Hatcher, *Algebraic Topology*; Bott & Tu, *Differential Forms in Algebraic Topology*.

Contents

1. INTRODUCTION

Algebraic topology concerns the connectivity properties of topological spaces. Recall that:

Definition 1.1. *A topological space X is* **connected** *if we cannot write $X = U \cup V$, where U, V are non-empty, open and disjoint subsets of X.*

Example 1.1. *$\mathbb{R}$ is connected (with its Euclidean topology) whilst $\mathbb{R}\backslash\{0\}$ is not.*

The first basic result we usually see about connectedness is:

Corollary 1.1 (Intermediate Value Theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) > 0$, $f(y) < 0$, then $\exists z$ between x and y such that $f(z) = 0$.*

Proof. If $f(z) \neq 0$ for all z, then $\mathbb{R} = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ is a disjoint union of non-empty open subsets of $\mathbb{R}$, which contradicts the fact that $\mathbb{R}$ is connected. ☐

For 'nice' spaces, connectedness is equivalent to path connectedness.

Definition 1.2. *A topological space X is* **path-connected** *if $\forall x, y \in X$, $\exists \gamma : [0,1] \to X$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Informally, this just means that any two maps of a point to X can be continuous deformed into one another (as we can just follow the path).

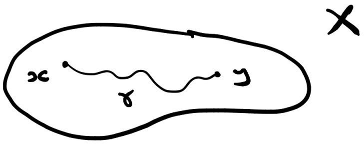


FIGURE 1. Path connectivity.

In this course, a **map** will always mean a continuous function.

Definition 1.3. *If X, Y are topological spaces and $f, g : X \to Y$ are maps, then f is* **homotopic** *to g if $\exists F : [0,1] \times X \to Y$ continuous such that $F|_{\{0\}\times X} = f$ and $F|_{\{1\}\times X} = g$.*

We write $f \simeq g$, or $f \underset{F}{\simeq} g$.

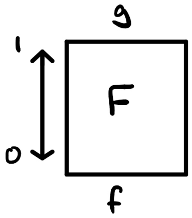


FIGURE 2. Schematic of a homotopy.

Note: So path connectedness just says that any two constant maps (or maps from a point into X) are homotopic.

Definition 1.4. *A path-connected space X is* ***simply connected*** *if any two continuous maps $S^1 \to X$ are homotopic*

i.e. if any two loops in X can be continuously deformed into one another (in X).

Here, $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is the n-dimensional sphere. So $S^1 =$ circle $\subset \mathbb{C}$.

Example 1.2. *$\mathbb{R}^2$ is simply connected, but $\mathbb{R}^2\backslash\{0\}$ is not. In fact, continuous maps $\gamma : S^1 \to \mathbb{R}^2\backslash\{0\}$ can be assigned a number $\deg(\gamma) \in \mathbb{Z}$, called the* ***degree*** *of γ, which turns out to be invariant under homotopy (this is just the* ***winding number*** *of γ).*

[If γ was differentiable we could set $\deg(\gamma) = \frac{1}{2\pi i}\int_\gamma \mathrm{d}z/z \in \mathbb{Z}$.]

So to see that $\mathbb{R}^2\backslash\{0\}$ is not simply connected, consider $\gamma_n : S^1 \to \mathbb{R}^2\backslash\{0\}$ defined by $t \mapsto e^{2\pi int}$. Then we can show that $\deg(\gamma_n) = n$, and so since the degree is homotopy invariant this shows that, e.g. $\gamma_1 \not\simeq \gamma_0$, and so $\mathbb{R}^2\backslash\{0\}$ is not simply connected.

A classical result which is a consequence of simply connectedness is the fundamental theorem of algebra.

Corollary 1.2 (Fundamental Theorem of Algebra)**.** *Every non-constant complex polynomial has a root.*

Proof. Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a complex polynomial, and suppose $f(z) \neq 0$ for all $z \in \mathbb{C}$. Then let $\gamma_R(t) := f\left(Re^{2\pi it}\right)$. So as $f \neq 0$, we have $\gamma_R : S^1 \to \mathbb{C}\backslash\{0\} \cong \mathbb{R}^2\backslash\{0\}$.

Clearly γ_0 is the constant map, and so $\gamma_0(0) = 0$. So hence by homotopy invariance of degree, $\deg(\gamma_R) = 0$ for all $R > 0$. [Indeed, the homotopy can be taken to be $F : [0,1] \times S^1 \to \mathbb{R}^2\backslash\{0\}$, $(\tau, t) \mapsto \gamma_{R\tau}(t)$].

But then if $R \gg \sum_i |a_i|$, we can consider $f_s(z) = z^n + s(a_1 z^{n-1} + \cdots + a_n)$ for $s \in [0,1]$. Then on the circle $t \mapsto Re^{2\pi it}$ we have $f_s(z) \neq 0$, and so f_s is also valued in $\mathbb{R}^2 \backslash \{0\}$ on this circle. So if

$$\gamma_{R,s}(t) := f_s(Re^{2\pi it})$$

then $\gamma_{R,1} = \gamma_1$, and clearly all $\gamma_{R,s}$ are homotopic for different s. But then $\gamma_{R,0} : z \mapsto z^n$ has degree n, and so by homotopy invariance of degree,

$$0 = \deg(\gamma_0) = \deg(\gamma_R) = \deg(\gamma_{R,1}) = \deg(\gamma_{R,0}) = n$$

i.e. $n = 0$ and so f must be constant.

So we have shown if f is never 0 it must be constant, and so hence if it is non-constant it must have a root.

□

Fact: Any two maps $S^n \to \mathbb{R}^{n+1}$ are homotopic. But maps $f : S^n \to \mathbb{R}^{n+1} \backslash \{0\}$ have a degree $\deg(f) \in \mathbb{Z}$, which is invariant under homotopy. Moreover, $\deg(\text{constant map}) = 0$ and $\deg(\text{inclusion map}) = 1$.

Corollary 1.3 (Brouwer's Fixed Point Theorem). *Let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Then any continuous map $f : B^n \to B^n$ has a fixed point.*

Proof. Suppose f has no fixed point. Then let $\gamma_R : S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ be the map $v \mapsto Rv - f(Rv)$, for $R \in [0,1]$. Note γ_R is valued in $\mathbb{R}^n \backslash \{0\}$ since f has no fixed points. Then clearly γ_0 is a constant, so $\deg(\gamma_0) = 0$. So by homotopy invariance (since $F(t,v) := \gamma_t(v)$ is continuous) we have $\deg(\gamma_1) = 0$.

Now let $\gamma_{1,s}(v) := v - sf(v)$, for $s \in [0,1]$ and $v \in S^{n-1}$. Note that $\gamma_{1,s}$ has image $\subset \mathbb{R}^n \backslash \{0\}$, since if $s = 1$ then this is because f has no fixed points, and if $s < 1$ then $1 = |v| > |sf(v)|$ for $v \in S^{n-1}$, since $|f(v)| \leq 1$ as f maps into B^n.

Therefore as all the $\gamma_{1,s}$ are homotopic and $\gamma_1 = \gamma_{1,1}$, we would have $\deg(\gamma_{1,0}) = \deg(\gamma_{1,1}) = \deg(\gamma_1) = 0$. But $\gamma_{1,0} : S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ is the inclusion map, which has degree 1. Hence we have a contradiction and so we are done.

□

Definition 1.5. *We say topological spaces X, Y are* ***homotopy equivalent*** *if $\exists$ maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.*

We write $X \simeq Y$.

Note: If X and Y are homeomorphic, i.e. $X \cong Y$, then clearly $X \simeq Y$. So homotopy equivalence is a weaker condition than homeomorphic.

Example 1.3.

- *$\mathbb{R}^n \simeq \{0\} \equiv$ a point; a space which is homotopy equivalent to a point is called* ***contractible****.*
- *$\mathbb{R}^n \backslash \{0\} \simeq S^{n-1}$. Indeed, if $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n \backslash \{0\}$ is the inclusion and $p : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ is the projection, $v \mapsto v/\|v\|$, then $p \circ \iota = \mathrm{id}_{S^{n-1}}$, and $\iota \circ p : v \mapsto v/\|v\|$, which is homotopy to $\mathrm{id}_{\mathbb{R}^n \backslash \{0\}}$ via the homotopy*
$$F : \mathbb{R}^n \backslash \{0\} \times [0,1] \to \mathbb{R}^n \backslash \{0\} \ \ , \ \ (v,t) \mapsto tv + (1-t)v/\|v\|.$$

Algebraic topology is just the study of {Topological Spaces}/Homotopy equivalence via looking at {Groups}/Isomorphism.

The first naive attempt to do this was via *homotopy groups*. Loops (by which we mean continuous maps $S^1 \to X$) with a common base point can be concatenated, and this induces a group structure on the set of homotopy classes of maps $(S^1, *) \to (X, x_0)$ [by which we mean continuous maps $S^1 \to X$ preserving the base point, i.e. $* \mapsto x_0$]. A **based homotopy** $F : f \simeq g$ of two such maps is a homotopy such that $F|_{S^1 \times \{t\}}$ sends $* \mapsto x_0$ for all t.

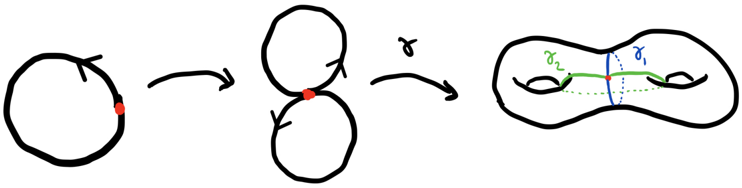


FIGURE 3. Illustration of circles with common base point and the different loops they can form in the image.

This group is what is known as the **fundamental group**, denoted $\pi_1(X, x_0)$. This leads to the generalisation S^n instead of S^1, where concatenation becomes one-point wedge product of n-spheres. Again, there is a group structure on the set of based homotopy classes of maps $(S^n, *) \to (X, x_0)$, denoted $\pi_n(X, x_0)$, the ***n*'th homotopy group** of X.

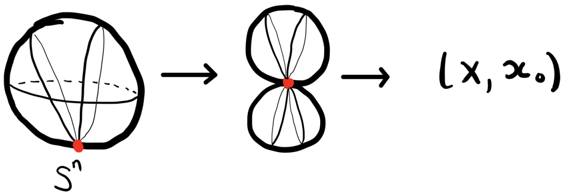


FIGURE 4. Wedge of two n-sphere's with common base point.

Fact: These homotopy groups are hard to compute - not even $\{\pi_n(S^2, x)\}_{n\geq 1}$ is known. Indeed, there is no simply connected manifold of dimension > 0 for which all π_n are known.

So as homotopy groups are hard to compute, we will instead focus on *homology* theory, and more precisely *singular (co)homology*. We will obtain invariants of spaces in a two-step process:

(i) Associate to X a *chain complex* (or *cochain complex*), done geometrically
(ii) Take (co)homology of that complex.

These will be rather computable for simple spaces. We will mostly focus on studying manifolds.

Definition 1.6. *A* ***chain complex*** (C_*, d) *is a sequence of abelian groups and homomorphisms*

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$$

(indexed by $\mathbb{N}$ *or* $\mathbb{Z}$*) such that* $d_{n-1} \circ d_n = 0$ *for all n.*

So the arrows go downward, decreasing the index, in a chain complex. Note that the condition $d_{n-1} \circ d_n = 0$ implies $\mathrm{Im}(d_n) \subset \ker(d_{n-1})$ are subgroups of C_{n-1}, and so we define:

Definition 1.7. *Given a chain complex* (C_*, d)*, the* ***n'th homology group*** $H_n(C_*, d)$ *is:*

$$H_n(C_*, d) := \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1})}$$

(i.e. a quotient of subgroups of C_n*).*

Similarly we can have a complex where the arrows go the other way, which is a *cochain complex*.

Definition 1.8. *A* ***cochain complex*** (C^*, d) *is a sequence of abelian groups and homomorphisms*

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots$$

such that $d^n \circ d^{n+1} = 0$ *for all n.*

The ***n'th cohomology group*** $H^n(C^*, d)$ *is:*

$$H^n(C^*, d) := \frac{\ker(d^n)}{\mathrm{Im}(d^{n-1})}.$$

2. SINGULAR (CO)CHAINS

We want to define a *simplex* in an arbitrary topological space. First we must define one in $\mathbb{R}^n$.

Definition 2.1. *A* ***n-simplex*** σ *in* $\mathbb{R}^{n+1}$ *is the convex hull of* $(n+1)$*—ordered points* $v_0, \ldots, v_n$ *in* $\mathbb{R}^{n+1}$ *such that* $\{v_i - v_0 : 1 \leq i \leq n\}$ *are linearly independent. We write*

$$\sigma = [v_0, \ldots, v_n].$$

The ***standard*** n***-simplex*** *is* $\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0\ \forall i\}$*, i.e. the convex hull of the standard basis of* $\mathbb{R}^{n+1}$*.*

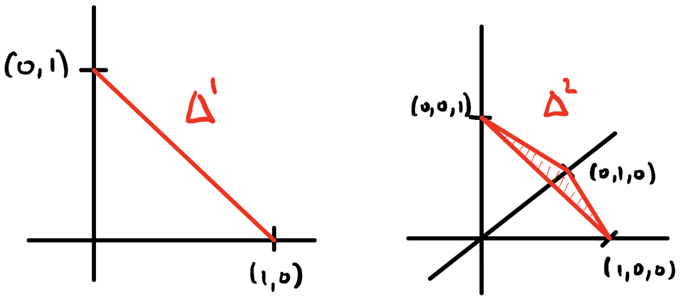


FIGURE 5. Illustrations of the Δ^1 and Δ^2.

Note: Any n-simplex in $\mathbb{R}^{n+1}$ is canonically the image of Δ^n under a linear homeomorphism $\Delta^n \to \sigma$, via $(t_i)_i \mapsto \sum_i t_i v_i \in \sigma$.

Definition 2.2. *An* n***-simplex*** *in a topological space* X *is a continuous map* $\sigma : \Delta^n \to X$ *(or from any n-simplex in to* X*).*

Note: Any n-simplex has faces, denoted $\Delta_i^{n-1} \subset \Delta^n$, defined by $\{t_i = 0\}$ (i.e. ignore the v_i direction). This then defines a corresponding face of any σ via the image of $\{t_i = 0\}$ (i.e. the face) under the map $\Delta^n \to \sigma$ above.

We write the i'th face of $\sigma = [v_0, \ldots, v_n]$ as: $[v_0, \ldots, \hat{v}_i, \ldots, v_n] \subset [v_0, \ldots, v_n]$, i.e. a hat over a vertex means we omit it.

The edges of any simplex are canonically oriented via "$v_i \to v_j$" if $i < j$.

Definition 2.3. *If* X *is a topological space, then the* ***singular chain complex*** $C_*(X;\mathbb{Z})$*, or just* $C_*(X)$*, is defined as follows. We have*

$$C_n(X) := \left\{ \sum_{i=1}^{N} h_i \sigma_i \ : \ N \in \mathbb{N}_{\geq 0},\ h_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \text{ is an } n\text{-simplex in } X \right\}$$

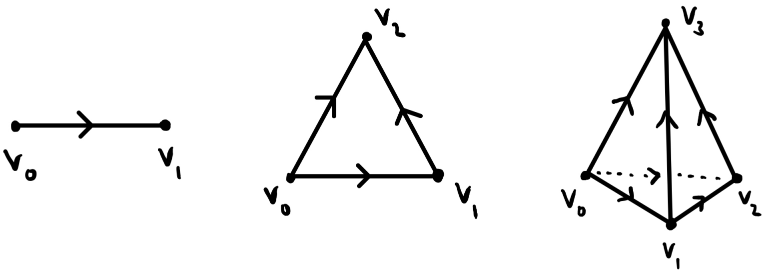


FIGURE 6. Orientated simplexes.

is the free abelian group on n-simplices in X, and the **boundary map** $d : C_n(X) \to C_{n-1}(X)$ *is defined by*

$$d\sigma := \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

where $\sigma = [v_0, \ldots, v_n]$*, and this is then extended linearly to all of* $C_n(X)$*.*

Example 2.1. *We have* $d([v_0, v_1, v_2]) = [v_0, v_1] - [v_0, v_2] + [v_1, v_2]$*.*

Lemma 2.1. $(C_*(X), \mathrm{d})$ *as above is indeed a chain complex, i.e.* $d^2 = d_{n-1} \circ d_n = 0$ *for all* $n \geq 1$*.*

Proof. We have

$$(d \circ d)(\sigma) = d\left(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\right) = \sum_{i=0}^{n} (-1)^i d\left(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\right).$$

Now when taking d again, we will either by removing a vertex before or after the one already removed, and so naturally we get two sums

$$= \sum_{j<i} (-1)^i \cdot (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} + \sum_{j>i} (-1)^i \cdot (-1)^{j-1} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]}$$

where the factor of $(-1)^{j-1}$ in the second sum comes from the fact that since v_i has been removed, when removing v_j, for $j > i$, this is the $(j-1)$'th vertex in $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$. So hence

$$(d \circ d)(\sigma) = \sum_{j<i} (-1)^{i+j} \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} - \sum_{j>i} (-1)^{i+j} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]}.$$

So noting that if we swap $i \longleftrightarrow j$ in the second sum we get the same as the first sum, and thus these two sums cancel. So $d^2 = 0$.

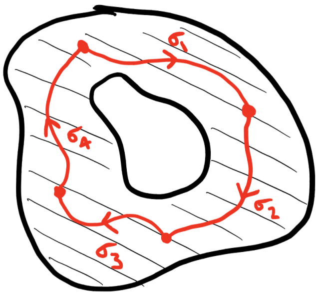
□

The resulting homology theory we get from this chain complex, denoted $H_*(X)$ or $H_*(X,\mathbb{Z})$, is called **singular homology**. The $\mathbb{Z}$ keeps track of the fact that we had $h_i \in \mathbb{Z}$, however we could similar define $C_*(X;G)$ and $H_*(X;G)$ for any abelian group G.

Note: To define $H_*(X;\mathbb{Z})$ we only used continuous maps into X, and thus $H_*(X;\mathbb{Z})$ only depends on the topology of X. Thus $H_*(X;\mathbb{Z})$ is tautologically a homeomorphism invariant of X.

So what is the intuitive picture behind the definition of the boundary map d above? THe idea is that d a region covered by simplices to its boundary, i.e.

$$d(\text{simplices}) = \text{boundary of covered region}.$$



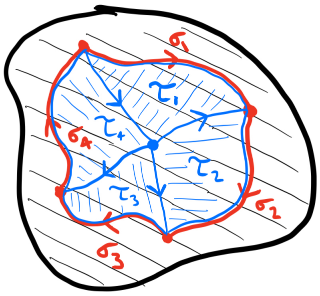
In the above diagram, we see that we have four simplices, $\sigma_1,\dots,\sigma_4$, which are line segments. Thus $d(\sigma_i)$ will give the boundary of σ_i, i.e. the difference between the endpoints of the line (so if $\sigma = [v_0,v_1]$, then $d(\sigma) = v_1 - v_0$). Thus we see that

$$d(\sigma_1+\sigma_2+\sigma_3+\sigma_4) = 0 \quad \text{i.e.} \quad \sigma_1+\sigma_2+\sigma_3+\sigma_4 \in \ker(d).$$

Motivated by this picture, we make the following definition.

Definition 2.4. *Elements of* $\ker(d : C_i(X) \to C_{i-1}(X))$ *are called* ***i-cycles****, or just* ***cycles****.*

Now consider the collection of simplices shown below.



Here we have 2-simplices $\tau_1,\dots,\tau_4$. When we consider $d(\tau_1+\tau_2+\tau_3+\tau_4)$, we are just left with $\sigma_1+\sigma_2+\sigma_3+\sigma_4$ (i.e. the boundary), since the internal edges connecting the outer vertices to the inner vertex cancel out in the alternating sum (i.e. they have 'opposite orientations' if you will). So hence

$$d(\tau_1+\tau_2+\tau_3+\tau_4) = \sigma_1+\sigma_2+\sigma_3+\sigma_4$$

and coupled with what we saw above, this shows that $\sigma_1+\sigma_2+\sigma_3+\sigma_4 \in \ker(d)\cap \mathrm{Im}(d)$. Once again, motivated by this picture we define:

Definition 2.5. *Elements of Im*$(d : C_i(X)\to C_{i-1}(X))$ *are called* ***boundaries****.*

So singular homology is cycles modulo boundaries.

Definition 2.6. *The* ***singular $\underline{\text{co}}$chain complex*** *of a space* X*, denoted* $C^*(X,\mathbb{Z})$ *or* $C^*(\mathbb{Z})$*, has* ***cochain groups***

$$C^n(X) := Hom(C_n(X),\mathbb{Z})$$

(i.e. the dual space of $C_n(X)$*), and* ***boundary maps*** $d^* : C^n(X)\to C^{n+1}(X)$ *defined by*

$$(d^*\psi)(\sigma) := \psi(d\sigma)$$

(i.e. usual dual map).

Observe that

$$(d^*(d^*\psi))(\sigma) = (d^*\psi)(d\sigma) = \psi(d^2\sigma) = \psi(0) = 0$$

and so $(d^*)^2 = 0$.

So indeed, $(C^*(X), d^*)$ $\underline{\text{is}}$ a cochain complex, simply because it is induced by a chain complex. The associated cohomology $H^*(X,\mathbb{Z})$ or $H^*(X)$ is called **singular cohomology**.

Note: $H^*(X,\mathbb{Z}) \neq \mathrm{Hom}_{\mathbb{Z}}(H_*(X,\mathbb{Z}),\mathbb{Z})$ in general (i.e. the cohomology group is not just the dual of the homology group).

Clearly if $f : X\to Y$ is continuous and $\sigma : \Delta^n \to X$ is a n-simplex in X, then we get an n-simplex in Y via $f\circ\sigma : \Delta^n\to Y$. Hence we get a map

$$f_* : C_*(X)\to C_*(Y) \quad \text{i.e.} \quad f_* : C_n(X)\to C_n(Y)\ \ \forall n$$

defined by

$$f_*\left(\sum_{i=1}^N h_i\sigma_i\right) := \sum_{i=1}^N h_i(f\circ\sigma_i)$$

which is clearly a group homomorphism. In particular, $f_*\sigma = f\circ\sigma$.

The key observation is that $df_* = f_*d$, since (by linearity suffices to check on a simplex σ)

$$\begin{aligned}(f_*d)(\sigma) &= f_*\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right)\\ &= \sum_{i=0}^n (-1)^i f\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right)\\ &= \sum_{i=0}^n (-1)^i (f\circ\sigma)|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\\ &= d(f_*\sigma)\end{aligned}$$

where the third line follows from the second just because faces are mapped to other faces by continuity.

This tells us that a continuous map $f : X \to Y$ induces a *chain map* of chain complexes, by which we mean:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "d"] \arrow[d, "f_*"] & C_n(X) \arrow[r, "d"] \arrow[d, "f_*"] & C_{n-1}(X) \arrow[r, "d"] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "d"] & C_n(Y) \arrow[r, "d"] & C_{n-1}(Y) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

such that all squares commute (and the f_* are group homomorphisms). In general:

Definition 2.7. *A* ***chain map*** *of chain complexes is a sequence of vertical maps between corresponding groups in the complexes which are group homomorphisms such that all squares commute.*

A simple algebraic result then is:

Lemma 2.2. *If C_* and D_* are chain complexes and $f_* : C_* \to D_*$ is a chain map, then f_* induces homomorphisms on homology, i.e. we get induced homomorphisms $f_* : H_i(C_*) \to H_i(D_*)$ for all i.*

Proof. Let $a \in H_i(C_*) = \ker(d_i : C_i \to C_{i-1})/\text{Im}(d : C_{i+1} \to C_i)$. So we know a is represented by some i-cycle $\alpha \in C_i$ with $d\alpha = 0$ (i.e. $a = [\alpha]$ is this equivalence class). Then

$$0 = f_*(d\alpha) = d(f_*\alpha)$$

i.e. $f_*(\alpha) \in \ker(d_i : D_i \to D_{i-1})$ is a cycle in the D_* chain complex, and so hence it defines an element $[f_*\alpha] \in H_i(D_*) = \ker(d : D_i \to D_{i-1})/\text{Im}(D_{i+1} \to D_i)$.

Set $b = [f_*\alpha]$ and define $f_* : H_i(C_*) \to H_i(D_*)$ by: $f_*(a) := b$, as above. Then we must show that this is well-defined and is a group homomorphism.

To see this is well-defined, suppose $a = [\alpha] = [\alpha']$, so that both α and α' are representatives of a. Then we know $[\alpha - \alpha'] = 0 \in H_i(C_*)$, i.e. $\alpha - \alpha'$ is a boundary, and so $\alpha - \alpha' = d_{i+1}(\gamma)$ for some $\gamma \in C_{i+1}$. Then:

$$f_*(\alpha) - f_*(\alpha') = f_*(\alpha - \alpha') = f_*(d\gamma) = d(f_*\gamma)$$

i.e. $f_*(\alpha)$ and $f_*(\alpha')$ differ by a boundary, and so $[f_*(\alpha)] = [f_*(\alpha')]$ in $H_i(D_*)$. So hence the image is independent of the choice of representative and so this map is well-defined.

To see this is a group homomorphism, suppose $a_1, a_2 \in H_i(C_*)$. Then if a_i is represented by $\alpha_i \in \ker(d)$, we have $\alpha_1 + \alpha_2$ represents $a_1 + a_2$. So hence as

$$[f_*(\alpha_1 + \alpha_2)] = [f_*(\alpha_1) + f_*(\alpha_2)] = [f_*(\alpha_1)] + [f_*(\alpha_2)]$$

this shows that it is a group homomorphism and so we are done.

□

The upshot is that if $f : X \to Y$ is a continuous map of topological spaces, then it induces a map on homology $f_* : H_i(X) \to H_i(Y)$, for each i, via

$$f_*([\alpha]) := [f_*(\alpha)].$$

Lemma 2.3. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps of topological spaces, with induced maps f_*, g_* on homology. Then:*

$$(g \circ f)_* = g_* \circ f_* \quad \textit{and} \quad (\mathrm{id})_* = \mathrm{id}.$$

Proof. We have

$$(g \circ f)_*([\alpha]) = [(g \circ f)_*(\alpha)] = [g(f(\alpha))] = g_*([f_*(\alpha)]) = g_*(f_*([\alpha]))$$

i.e. $(g \circ f)_* = g_* \circ f_*$.

Also,

$$\mathrm{id}_*([\alpha]) = [\mathrm{id}(\alpha)] = [\alpha]$$

i.e. $\mathrm{id}_* = \mathrm{id}$.

□

In category-theoretic language, the association $X \mapsto H_*(X)$ is a *functor* from the category of topological spaces (and homeomorphisms) to the category of graded abelian groups (and graded isomorphisms).

Note: If $f : X \to Y$ induces $f_* : C_*(X) \to C_*(Y)$, then this has an adjoint map $f^* : C^*(Y) \to C^*(X)$ on the cochain complex. This again induces a homomorphism on cohomology groups

$$f^* : H^*(Y) \to H^*(X).$$

Note that this homomorphism goes 'the other way', from cohomology of Y to that of X. [**Exercise** to check the details.]

2.1. **First Computations.**

Lemma 2.4. *We have*

$$H_*(\text{point}) = \begin{cases} \mathbb{Z} & \textit{if } * = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For each $n \geq 0$, there is a unique n-simplex in $X = \{\text{point}\}$, namely the constant map $\sigma_n : \Delta^n \to \{\text{point}\}$. So the chain complex $(C_*(\{\text{point}\}), d)$ is as follows:

$$\begin{tikzcd} \cdots \arrow[r] & \underset{\cong \mathbb{Z}}{C_n(X)} \arrow[r] & \underset{\cong \mathbb{Z}}{C_{n-1}(X)} \arrow[r] & \cdots \arrow[r] & \underset{\cong \mathbb{Z}}{C_2(X)} \arrow[r] & \underset{\cong \mathbb{Z}}{C_1(X)} \arrow[r] & \underset{\cong \mathbb{Z}}{C_0(X)} \end{tikzcd}$$

where each group is isomorphic to $\mathbb{Z}$ since they are the $\mathbb{Z}$-free group over a point/one element.

So in general we have $C_n(X) = \sigma_n\mathbb{Z}$, i.e. generated by the single n-simplex as above. So we have

$$d(\sigma_n) = \sum_{i=0}^{n}(-1)^i \underbrace{\sigma_n|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}}_{=\sigma_{n-1}\text{ always}} = \sigma_{n-1}\sum_{i=0}^{n}(-1)^i = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sigma_{n-1} & \text{if } n \text{ is even.}\end{cases}$$

So in terms of the $\mathbb{Z}$ groups, the maps are either the identity or the 0-map. So hence the above chain complex becomes

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}\ .$$

Hence we see

$$H_0(\{\text{point}\} = \frac{C_0(\{\text{point}\})}{\text{Im}(d_1 : C_1 \to C_0)} \cong \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z}$$

and if $i \geq 1$, then either $\ker(d_i) = \{0\}$ or d_{i+1} is surjective (so $\cong \mathbb{Z}$) and so

$$H_i(\{\text{point}\}) = \frac{\ker(d_i : C_i(X) \to C_{i-1}(X))}{\text{Im}(d_{i+1} : C_{i+1} \to C_i)} \cong \begin{cases} \frac{\{0\}}{\mathbb{Z}} & \text{if } n \text{ is odd} \\ \frac{\mathbb{Z}}{\mathbb{Z}} & \text{if } n \text{ is even}\end{cases} \cong \{0\}.$$

□

Lemma 2.5. *For any topological space X, $H_0(X)$ is the free abelian group generated by the set of path-components of X.*

Remark: The set of path-components of X is often written as $\pi_0(X)$ (the 0'th homology group). So this result tells us:

$$H_0(X) = \bigoplus_{\alpha\in\pi_0(X)} \mathbb{Z}.$$

Proof. We know that $X = \amalg_\alpha X_\alpha$ is a disjoint union of path-components of X_α.

Now any simplex $\sigma : \Delta^i \to X$ must have (by continuity) image inside a single X_α, and then the faces of σ have image in the same X_α. From this we see

$$(C_*(X), d) = \bigoplus_\alpha (C_*(X_\alpha), d_\alpha)$$

just because $C_*(X)$ is the free abelian group (over $\mathbb{Z}$) generated by the simplices in X, and thus we can decompose each such sum into sums over simplicies in each X_α.

Hence it suffices to prove that $H_0(X) \cong \mathbb{Z}$ if X is path connected.

So assume X is path-connected, and define $\varphi : C_0(X) \to \mathbb{Z}$ by:

$$\sum_i n_i\sigma_i \longmapsto \sum_i n_i,$$

where σ_i is a 0-simplex ($\equiv$ point) in X. Thus clearly if $X \neq \emptyset$, then φ is onto.

Now suppose τ is a 1-simplex in X. Then $d\tau = v_1 - v_0$ (the endpoints of τ), and so $\varphi(d\tau) = 1-1 = 0$. So by linearity we see that $\text{Im}(d : C_1 \to C_0) \subset \ker(\varphi)$.

Now suppose conversely that $\sum_{\text{finite}} n_i\sigma_i \in \ker(\sigma)$. Then fix a base point $p \in X$, and for each i choose a path ($\equiv$ 1-simplex) $\tau_i : [0,1] \to X$ such that $\tau_i(0) = p$ and $\tau_i(1) = \sigma_i$. Then

$$d\left(\sum_{\text{finite}} n_i\tau_i\right) = \sum_i n_i(d\tau_i) = \sum_i n_i\sigma_i - \underbrace{\left(\sum_i n_i\right)}_{=0} p = \sum_i n_i\sigma_i$$

since $\sum_i n_i\sigma_i \in \ker(\sigma)$. So hence we see $\sum_i n_i\sigma_i \in \mathrm{Im}(d : C_1 \to C_0)$, and so $\ker(\varphi) \subset \mathrm{Im}(d : C_1 \to C_0)$.

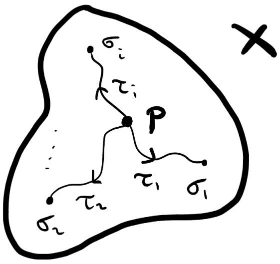


FIGURE 7. An illustration of the τ_i maps.

Hence we see $\ker(\varphi) = \mathrm{Im}(d : C_1 \to C_0)$, and so the first isomorphism theorem applied to φ gives

$$\mathbb{Z} = \mathrm{Im}(\varphi) \cong \frac{C_0(X)}{\ker(\varphi)} \cong \frac{C_0(X)}{\mathrm{Im}(d : C_1 \to C_0)} =: H_0(X)$$

as required. ☐

Remark: If X is path-connected, then we see that in fact $H_0(X) \cong \mathbb{Z}$ is generated by a point (any point in fact) of X.

As an informal conjecture, it turns out for reasonable spaces, we cannot compute anything else directly from the definition. For example, for manifolds of dimension > 0, each $C_i(X)$ is uncountably generated. This makes it very hard to work from the defintion.

So instead, we need other ways to compute homology. The tools developed are used to compute the homology of a more complicated space from the homology of small spaces which the bigger space can be decomposed into, or are homotopic to.

Digression. Suppose X is compact and Y has a metrisable topology, and pick a metric d_Y on Y which induces the topology on Y. Then Maps(X,Y), the set of continuous functions $X \to Y$, inherits a metric via:

$$d(f,g) := \sup_{x\in X} d_Y(f(x), g(x)).$$

[This is the **compact-open topology**. In fact, the resulting topology on Maps(X,Y) is independent of the choice of d_Y.]

A **knot** is an embedding $S^1 \hookrightarrow S^3$. Most of classical knot theory is computing $H_0(\mathrm{Emb}(S^1,S^3))$. If M is simply connected and $\dim(M) \geq 4$, then $H_*(\mathrm{Homeo}(M))$ (homeomorphisms $M \to M$) is unknown. The point is that computing homology groups can still be hard, and is a major research area in maths.

However homology is rendered (moderately) computable via the following two results:

Theorem 2.1 (Homotopy Induces Maps on Homology). *Suppose $f, g : X \to Y$ are homotopic maps of topological spaces X, Y. Then we have $f_* = g_*$ as maps $H_*(X) \to H_*(Y)$, and similarly $f^* = g^*$ as maps $H^*(Y) \to H^*(X)$.*

So this result just says that homotopic maps induce the same maps on homology and cohomology. This essentially says that homology is “insensitive to inessential” deformations of a space.

Proof. Later. □

Corollary 2.1 (Homology is invariant under Homotopy Equivalence).

Let X, Y be topological spaces which are homotopy equivalent. Then we have $H_*(X) \cong H_*(Y)$ and $H^*(X) \cong H^*(Y)$.

Proof. By definition of homotopy equivalence, we know ∃ maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. Hence by Theorem 2.1,

$$f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{H_*(Y)} \quad \text{and similarly} \quad g_* \circ f_* = \mathrm{id}_{H_*(X)}$$

where $f_* : H_*(X) \to H_*(Y)$ and $g_* : H_*(Y) \to H_*(X)$. Hence these maps are bijections and homomorphisms, and thus we get $H_*(X) \cong H_*(Y)$.

We can do the same thing for $H^*(X) \cong H^*(Y)$.

□

Corollary 2.2. *We have*

$$H_*(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \textit{if } * = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We know that $\mathbb{R}^n \simeq \{\text{point}\}$ are homotopy equivalent. Hence the result follows from Corollary 2.1 and Lemma 2.4.

□

So homotopy equivalence preserving homology is one way we can compute homology of more complicated spaces. Another is the *Mayer-Vietoris Property*, which enables us to compute the homology of a space by decomposing it into smaller parts.

Theorem 2.2 (Mayer-Vietoris (MV))**.** *Let $X = A \cup B$ be a union of open subsets. Note that we have a natural diagram of maps*

$$\begin{tikzcd}
A \cap B \arrow[r, hook, "i_A"] \arrow[d, hook, "i_B"] & A \arrow[d, hook, "j_A"] \\
B \arrow[r, hook, "j_B"] & X
\end{tikzcd}$$

of inclusion maps. Then $\exists$ boundary maps, called ***Mayer-Vietoris boundary maps****, $\partial_{MV} : H_i(X) \to H_{i-1}(A \cap B)$ for all $i \geq 1$, such that the sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(A \cap B) \arrow[r, "{(i_{A_*}, i_{B_*})}"] & H_{i+1}(A) \oplus H_i(B) \arrow[r, "j_{A*} - j_{B_*}"] & H_{i+1}(X) \arrow[dll, rounded corners, "\partial_{MV}"] \\
 & H_i(A \cap B) \arrow[r, "{(i_{A_*}, i_{B_*})}"] & H_i(A) \oplus H_i(B) \arrow[r, "j_{A_*} - j_{B_*}"] & \cdots
\end{tikzcd}$$

is exact.

Proof. Later. ☐

Definition 2.8. *A (co)chain complex is* ***exact*** *if it has zero (co)homology,*

i.e. if $\ker(d_i) = \text{Im}(d_{i+1})$ $\forall i$ *for homology (and similar for cohomology).*

We then have some addenda/consequences to MV:

(i) We have Mayer-Vietoris on cohomology; i.e. $\exists$ maps $\partial^*_{MV} : H^i(A \cap B) \to H^i(X)$ such that

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^i(A) \oplus H^i(B) \arrow[r, "i_A^* - i_B^*"] & \mathrm{H}^i(A \cap B) \arrow[r, "\partial^*_{MV}"] & \mathrm{H}^i(X) \arrow[dll, rounded corners, "(j_A^* j_B^*)"] \\
 & \mathrm{H}^{i+1}(A) \oplus H^{i+1}(B) \arrow[r, "i_A^* - i_B^*"] & \mathrm{H}^{i+1}(A \cap B) \arrow[r, "\partial^*_{MV}"] & \cdots
\end{tikzcd}$$

is exact.

(ii) The MV sequences are **natural**: by this we mean that if $X = A \cup B$ and $Y = C \cup D$, and $f : X \to Y$ is such that $f(A) \subset C$ and $f(B) \subset D$, then we get a map of MV sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(X) \arrow[r, "\partial_{MV}"] \arrow[d, "f_*"] & H_i(A \cap B) \arrow[r] \arrow[d, "f_*"] & H_i(A) \oplus H_i(B) \arrow[r] \arrow[d, "f_*"] & H_i(X) \arrow[r, "\partial_{MV}"] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_{i+1}(Y) \arrow[r, "\partial_{MV}"] & H_i(C \cap D) \arrow[r] & H_i(C) \oplus H_i(D) \arrow[r] & H_i(Y) \arrow[r, "\partial_{MV}"] & \cdots
\end{tikzcd}$$

and all squares commute.

We similarly have such a naturallity property for cohomology MV, with f^*, etc.

(iii) (What the MV map does) Suppose $Z \in H_n(X)$ is represented by an n-cycle of the form $z = a + b$, with $a \in C_n(A)$, $b \in C_n(B)$ (n-chains). So:

$$dz = 0 \quad \Rightarrow \quad da = -db$$

and so as $da \in C_{n-1}(A)$, $db \in C_{n-1}(B)$, we see that $da \in C_{n-1}(A) \cap C_{n-1}(B) = C_{n-1}(A \cap B)$. So hence as $d^2 = 0$, we see that da defines an element in $H_{n-1}(A \cap B)$, i.e. $[da] \in H_{n-1}(A \cap B)$ is defined.

Then ∂_{MV} is defined by:

$$\partial_{MV}(z) := [da] \quad \text{i.e.} \quad \partial_{MV}(a+b) := [da] = [db] \in H_{n-1}(A \cap B).$$

Note: At the moment, we haven't justified that $z = a + b$!

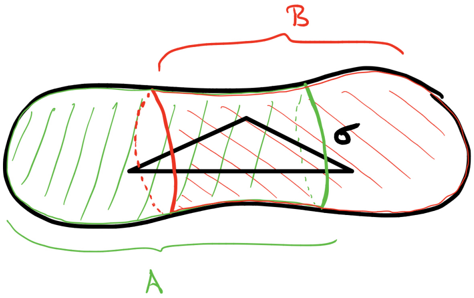


FIGURE 8. Illustration of dividing a space.

We will come back and prove homotopy equivalence invariance of homology and MV later. For now we will use them to see what we can prove, in the spirit of if you get a new toy, you first play with it before taking it apart to see how it works.

Lemma 2.6. *We have*

$$H_*(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can write $S^1 = X = A \cup B$, where A, B are open intervals, as shown.

Hence we clearly have $A, B \simeq \{\text{point}\}$, and $A \cap B$ is a union of two open intervals, and so $A \cap B \simeq \{\text{point}\} \amalg \{\text{point}\} \simeq \{p\} \amalg \{q\}$, as shown.

Homotopy invariance of homology then implies that $H_*(A)$, $H_*(B)$ and $H_*(A \cap B)$ are only non-zero at $* = 0$. So the MV sequence for $i \geq 2$ gives:

$$\underbrace{H_i(A) \oplus H_i(B)}_{\{0\} \oplus \{0\} = \{0\}} \longrightarrow H_i(S^1) \longrightarrow \underbrace{H_{i-1}(A \cap B)}_{=0}$$

and this sequence is exact. Hence this implies $H_i(S^1) = \{0\}$ for $i \geq 2$.

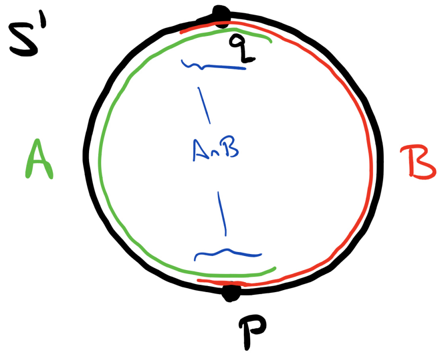


FIGURE 9. Computing homology of S^1.

For $i = 1$, the MV sequence gives exactness of:

$$\underbrace{H_1(A\cap B)}_{=0} \longrightarrow \underbrace{H_1(A)\oplus H_1(B)}_{=0} \longrightarrow H_1(S^1) \xrightarrow{\beta} \underbrace{H_0(A\cap B)}_{\cong \mathbb{Z}\langle p\rangle \oplus \mathbb{Z}\langle q\rangle} \xrightarrow{\alpha} \underbrace{H_0(A)\oplus H_0(B)}_{\cong \mathbb{Z}\oplus\mathbb{Z}} \longrightarrow H_0(S^1).$$

Note that we know $H_0(S^1) = \mathbb{Z}$ by Lemma 2.5 since S^1 is path-connected.

In the above sequence, $\alpha = (i_{A*}, i_{B^*})$, and so we have (since p, q are in both A, B)

$$\alpha(m,n) = (m+n, m+n)$$

where by (m,n) we mean m copies of the point p and n copies of q, and so the total number of points in A/B is $m+n$ (just from what i_{A*}, etc, are).

Exactness of the sequence at $H_0(A\cap B)$ gives $\ker(\alpha) = \text{Im}(\beta)$ and exactness at $H_1(S^1)$ gives $\ker(\beta) = \{0\}$. So β is injective, and so by the first isomorphism theorem,

$$H_1(S^1) \cong \text{Im}(\beta) = \ker(\alpha) = \mathbb{Z}\langle(1,-1)\rangle \cong \mathbb{Z}\langle p-q\rangle \cong \mathbb{Z}$$

and so we are done.

□

Note: This does give an explicit generator for $H_1(S^1)$, namely $p-q$. This will be useful in the future.

Exercise: Show similarly using the MV sequence for cohomology that

$$H^*(S^1) = \begin{cases} \mathbb{Z} & \text{if } * = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.7 ((Co)Homology of S^n)**.** *For $n \geq 1$ we have*

$$H_*(S^n) = \begin{cases} \mathbb{Z} & \text{if } * = 0,n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H^*(S^n) = \begin{cases} \mathbb{Z} & \text{if } * = 0,n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For a bit of variety, we will show the cohomology calculation this time. We will prove this by induction on n. We have already proven the $n=1$ case.

Write $S^n = A \cup B$, where A, B are the open hemispheres defined by $x_n \geq -\varepsilon$ or $x_n < \varepsilon$ respectively, for some $\varepsilon > 0$. So hence as these are hemispheres we have $A, B \simeq \{\text{point}\}$ and $A \cap B \simeq S^n \cap \{x_n = 0\} \simeq S^{n-1}$.

Then consider the MV sequence, which says the following sequence is exact:

$$H^i(S^n) \longrightarrow H^i(A) \oplus H^i(B) \longrightarrow H^i(A \cap B) \longrightarrow H^{i+1}(S^n) \longrightarrow H^{i+1}(A) \oplus H^{i+1}(B).$$

If $i \geq 1$, this then becomes:

$$0 \longrightarrow H^i(S^{n-1}) \longrightarrow H^{i+1}(S^n) \longrightarrow 0$$

is exact, i.e. $H^i(S^{n-1}) \cong H^{i+1}(S^n)$ for all $i \geq 1$. Hence by induction we have found $H^i(S^n)$ for all $i \geq 2$.

If $i = 0$, then this becomes

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \underbrace{H^0(S^{n-1})}_{\cong \mathbb{Z} \text{ by induction}} \xrightarrow{\beta} H^1(S^n) \longrightarrow 0$$

where the map α is $(p, q) \mapsto p - q$. Clearly this map is surjective. Hence by exactness, we see that $\ker(\beta) = \operatorname{Im}(\alpha) = \mathbb{Z}$. Hence β is the zero map, and so by exactness at $H^1(S^n)$ we then get $H^1(S^n) = \operatorname{Im}(\beta) = \{0\}$. Hence by induction, $H^1(S^n) = \{0\}$ for all $n \geq 2$.

Clearly by path-connectedness and Lemma 2.5 (the corresponding result for cohomology) we know that $H^0(S^1) = \mathbb{Z}$, and so we are done.

□

Corollary 2.3 (Topology sees Dimension). *We have*

$$\mathbb{R}^m \cong \mathbb{R}^n \text{ are homeomorphic} \iff m = n.$$

Proof. (⇐): Clearly true (take the identity map).

(⇒): Suppose $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. Then it induces a homeomorphism $\mathbb{R}^m \backslash \{0\} \to \mathbb{R}^n \backslash \{\varphi(0)\}$. So hence as $\mathbb{R}^k \backslash \{\text{point}\} \cong S^{k-1}$ in general, we see that this gives a homeomorphism $S^{m-1} \to S^{n-1}$. Hence these spheres are homotopy equivalent, and so by Corollary 2.1 we have $H_*(S^{m-1}) \cong H_*(S^{n-1})$ for all $*$. Then from Lemma 2.7 we see this implies $m = n$.

□

2.1.1. *Order of maps $S^n \to S^n$.*

Suppose $f : S^n \to S^n$ is a continuous map. Then it induces (in the usual way) a homomorphism $f_* : H_*(S^n) \to H_*(S^n)$. Clearly the only map of interest here (as either the groups are zero, or at $* = 0$ the map is just the identity since it maps a point to a point) if $f_* : H_n(S^n) \to H_n(S^n)$, since $H_n(S^n) \cong \mathbb{Z}$, we see that we get a homomorphism $f_* : \mathbb{Z} \to \mathbb{Z}$. Hence f_* must be multiplication by some integer. We then define:

Definition 2.9. *For $f : S^n \to S^n$ continuous, we define the* ***degree of f*** *by the integer* $\deg(f) \in \mathbb{Z}$ *such that $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by* $\deg(f)$.

This degree is well-defined if we use the same isomorphism $H_n(S^n) \to \mathbb{Z}$. Equivalently, since $\mathrm{Im}(f_*) = \deg(f)\mathbb{Z}$ we have

$$\deg(f) = \left| \frac{\mathbb{Z}}{\mathrm{Im}(f_*)} \right|.$$

Lemma 2.8 (Properties of $\deg(f)$). *We have:*

- *(i)* $\deg(f \circ g) = \deg(f)\deg(g)$,
- *(ii)* $\deg(\mathrm{id}) = 1$,
- *(iii)* $\deg(\text{constant map}) = 0$.

Proof. (i): This is simply because $(f \circ g)_* = f_* \circ g_*$, and so

$$\deg(f \circ g) = (f \circ g)_*(1) = f_*(g_*(1)) = f_*(\deg(g)) = \deg(f)\deg(g).$$

(ii): Simply because $\mathrm{id}_* = \mathrm{id}$.

(iii): If $\varphi : S^n \to S^n$ is constant, then we can write $\varphi = \iota \circ \tilde{\varphi}$, where $\tilde{\varphi} : S^{n-1} \to \{\varphi(1)\}$ is the constant map and $\iota : \{\varphi(1)\} \hookrightarrow S^{n-1}$ is the inclusion. This is just saying that φ factors through

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[rr, bend right, "\varphi"'] & \{\text{point}\} \arrow[r] & S^{n-1} \end{tikzcd}$$

So hence taking $*$ of this diagram we see that φ_* factors through $H_n(\{\text{point}\}) = \{0\}$ (as $n > 0$), and hence $\deg(\varphi) = 0$.

□

Lemma 2.9. *Suppose $A \in O(n+1)$. Then $A : S^n \to S^n$ acts on $H_n(S^n)$ by multiplication by* $\det(A)$, *i.e.*

$$\deg(A) = \det(A) \quad (= \pm 1).$$

Proof. $O(n+1)$ has two connected components, distinguished by the sign of the determinant (± 1).

For the $\{\det(A) = +1\}$ component, we have $A \simeq I \in O(n+1)$, and so $A_* = \mathrm{id}_* = id$ is the identity map. Hence A_* is multiplication by $\deg(A) = \deg(\mathrm{id}) = 1 = \det(A)$, and so in this case we are done.

So it suffices to show that if A is a reflection in a hyperplane (i.e. $\det(A) = -1$) then $\deg(A) = -1$. So let A be a reflection in the hyperplane H, and write $A = \mathrm{refl}_H$. Then notice that the hyperplane will divide S^n into two equal hemispheres, both of which are invariant under the action of refl_H. refl_H also induces a reflection on the middle line ∂L via reflection in the plane $H' = H \cap \pi(\partial L)$ (projection onto H), which is $\simeq S^{n-1}$ since in some basis this is equivalent to having $\{x^n = 0\} \cap S^n \simeq S^{n-1}$.

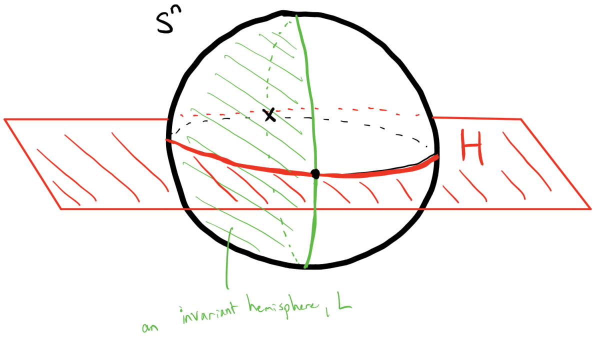


FIGURE 10. An illustration of the reflection map and application of MV. The two invariant hemispheres give rise to an S^{n-1} via their common boundary, shown in green.

So hence applying Mayer-Vietoris to the two (closed) invariant hemispheres whose intersection is $H' \simeq S^{n-1}$, we get a diagram (recall the addenda of MV)

$$\begin{tikzcd}
0 \arrow[r] & H_n(S^n) \arrow[r, "\cong"] \arrow[d, "(\mathrm{refl}_H)_*"] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "(\mathrm{refl}_H)_*"] & 0 \\
0 \arrow[r] & H_n(S^n) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

and naturality of Mayer-Vietoris tells us that this diagram commutes.

Then by induction on this diagram, it shows that it is sufficient to prove the $n = 1$ case.

So consider the $n = 1$ case, shown in Figure 11. Recall that when we compute $H_1(S^1) \cong \mathbb{Z}$, we found that the generator was $\langle p - q \rangle$. But then reflection in H swaps p and q, and so gives the generator $\langle q - p \rangle = \langle -(p - q) \rangle$, i.e. this acts by -1.

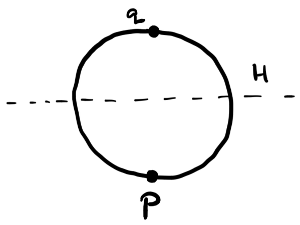


FIGURE 11. The $n = 1$ case.

□

Corollary 2.4. *We have*

- *(i) The antipodal map $a_n : S^n \to S^n$, $a_n(x) := -x$, has degree $(-1)^{n+1}$.*
- *(ii) If $f : S^n \to S^n$ has no fixed point, then $f \simeq a_n$.*
- *(iii) If G acts freely on S^{2k} then $G \leq \mathbb{Z}_2$.*
- *(iv)* [Hairy-Ball Theorem]

$$S^n \textit{ has a nowhere-zero vector field } \iff n \textit{ is odd.}$$

Proof. <u>(i):</u> a_n is the composite of $(n+1)$—reflections (the coordinate axes), and thus Lemma 2.9 and Lemma 2.8(i),

$$\deg(a_n) = \underbrace{(-1)\cdot(-1)\cdots(-1)}_{(n+1)\text{—times}} = (-1)^{n+1}.$$

<u>(ii):</u> In fact we will prove a stronger result, which says that if $f, g : S^n \to S^n$ have $f(x) \neq g(x)$ for all $x \in S^n$, then $f \simeq a_n \circ g$.

Indeed, consider the map $\varphi_t : S^n \to S^n$ for $t \in [0,1]$:

$$x \longmapsto \frac{tf(x)-(1-t)g(x)}{\|tf(x)-(1-t)g(x)\|}.$$

Note that this is well-defined, since the denominator never vanishes. Indeed, if $t \neq 1/2$, this is because if we ever had $tf(x) = (1-t)g(x)$, then taking modulus', since $f(x), g(x) \in S^n$, we would have $t = 1-t$, i.e. $t = 1/2$, a contradiction. However when $t = 1/2$, this never vanishes since $f(x) \neq g(x)$ for all $x \in S^n$.

Thus we have $f = \varphi_1 \simeq \varphi_0 = a_n \circ g$.

The result we are after follows by taking $g = \text{id}_{S^n}$.

<u>(iii):</u> Suppose G acts freely on S^{2k}. Then for all $g \in G\backslash\{e\}$, g has no fixed point and thus by (ii) we know $g \simeq a_{2k}$, and so by (i) we have $\deg(g) = -1$. Thus if we define a map $F : G \to \mathbb{Z}_2$ by $F(g) := \deg(g)(= -1)$, then this is a homomorphism (by Lemma 2.8) and has no kernel by the above (as nothing maps to +1), and thus is injective. Hence by the 1st isomorphism theorem we have

$$G = \frac{G}{\ker(F)} \cong \text{Im}(F) \leq \mathbb{Z}_2.$$

[Contrast this with S^1, which acts freely on $S^{2n+1} \subset \mathbb{C}^{k+1}$ by rotation.]

<u>(iv):</u> A vector field on S^n is a continuous map $v : S^n \to \mathbb{R}^{n+1}$ such that for all $x \in S^n$ we have $\langle x, v(x)\rangle = 0$ (Euclidean inner product on $\mathbb{R}^{n+1}$).

<u>($\Leftarrow$):</u> If n is odd, we can write down such a vector field explicitly. Indeed, define

$$v(x_0, y_0, \ldots, x_k, y_k) := (y_0, -x_0, y_1, -x_1, \ldots, y_k, -x_k)$$

where $n+1 = 2k$. Clearly v has $\langle x, v(x)\rangle = 0$ for all $x \in S^n$, so this is an example.

(⇒): Suppose a nowhere zero vector field on S^n exists, and call it v. Then consider the map $F : S^n \to S^n$ defined by $F(x) = v(x)/\|v(x)\|$. Then consider the family of maps $\varphi_t : S^n \to S^n$ for $t \in [0,1]$ defined by

$$\varphi_t(x) := (\cos(t))x + \sin(t)F(x).$$

Hence $\mathrm{id}_{S^n} = \varphi_0 \simeq \varphi_\pi = a_n$, the antipodal map. So hence

$$(-1)^{n+1} = \deg(a_n) = \deg(\mathrm{id}_{S^n}) = 1 \quad \Longrightarrow \quad n \text{ is odd.}$$

□

Now we finish with one final calculation before going back to prove the homotopy invariance of homology and Mayer-Vietoris.

Lemma 2.10. *Let K be a Klein bottle. Then:*

$$H_*(K;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We know that $K = \text{Möb} \cup_\partial \text{Möb}$, where Möb are Möbius strips and the union means we glue them together along a common boundary. Thus we can write $K = A \cup B$, where A, B are the Möbius strips plus an ‘extra bit’ to make them overlap. Thus we have $A, B \simeq S^1$ and $A \cap B \simeq$ boundary of a Möbius strip $\simeq S^1$. The situation is as in Figure 12.

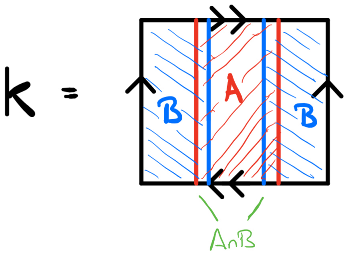


FIGURE 12. An illustration of the Klein bottle, with the sets A, B identified. Notice that the sides of the square are identified in the usual way, and thus A, B are both Möbius strips. Due to the ‘opposite orientation’ being used on the top and bottom sides, when attaching them to one another this causes the side to flip, and thus the LHS of the top side will attach to the RHS of the bottom side, etc, and thus after this identification we see that $A \cap B$ is one connected component and is $\simeq S^1$.

So applying Mayer-Vietoris gives

$$0 \to H_2(K) \to H_1(A \cap B) \xrightarrow{\psi} H_1(A) \oplus H_1(B) \to H_1(K) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B).$$

Filling in the groups we know from the identifications of $A, B, A \cap B$ with S^1, we have

$$0 \longrightarrow H_2(K) \longrightarrow \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(K) \longrightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \oplus \mathbb{Z}.$$

So what are the maps ψ, φ? φ is induced from the inclusion of $A \cap B$ into A and B, mapping points to points. Hence on $A \cap B \to A$, this is just $p \mapsto p$, and on B this map is the same, and so $H_0(A \cap B) \to H_0(A) \oplus H_0(B)$ is given by $p \mapsto (p, p)$, i.e. in terms of $\mathbb{Z}$, this is $\varphi : 1 \mapsto (1, 1)$.

For the ψ map, the maps are induced by the inclusions of circles in $A \cap B$ into A or B. Let us draw what is happening to work out what ψ is.

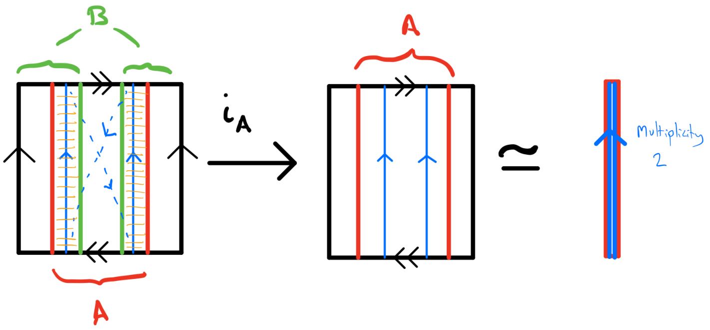


FIGURE 13. An illustration of calculating the ψ map. We have a circle in $A \cap B$, which is the two strips shown. The two blue lines representing the circle are in fact just 1 S^1 due to the opposite orientation on the top and bottom sides - it just 'wrap around' twice before getting back to the start. When including this into A (say), we get left with the same two lines. Then when we identify $A \simeq S^1$ both lines become one and overlap each other - which gives a circle with multiplicity 2. Hence we have one circle becoming two, and thus $(i_A)_*(1) = 2$. But the same thing can be said for B, and thus we get $(i_B)_*(1) = 2$, and so $\psi(1) = (2, 2)$.

Thus we see $\psi(1) = (2, 2)$ and with this we can calculate the homology groups. We therefore have

$$\ker(\varphi) = \{0\}, \quad \mathrm{Im}(\varphi) \cong \mathbb{Z}, \quad \ker(\psi) = \{0\}, \quad \mathrm{Im}(\psi) \cong 2\mathbb{Z}.$$

The first isomorphism theorem at $H_2(K)$ gives

$$H_2(K)/\ker(H_2(K) \to \mathbb{Z}) \cong \mathrm{Im}(H_2(K) \to \mathbb{Z})$$

and so using exactness we get

$$H_2(K) \cong H_2(K)/\mathrm{Im}(0 \to H_2(K)) \cong H_2(K)/\ker(H_2(K) \to \mathbb{Z}) \cong \mathrm{Im}(H_2(K) \to \mathbb{Z}) \cong \ker(\psi) = \{0\}.$$

Then using exactness at the $\mathbb{Z}$ after $H_1(K)$ we get

$$\mathrm{Im}(H_1(K) \to \mathbb{Z}) = \ker(\varphi) = \{0\}$$

and thus the isomorphism theorem gives

$$H_1(K)/\ker(H_1(K) \to \mathbb{Z}) \cong \mathrm{Im}(H_1(K) \to \mathbb{Z}) = \{0\}$$

and so by exactness and the first isomorphism theorem we have

$$\begin{aligned} H_1(K) \cong \ker(H_1(K) \to \mathbb{Z}) &\cong \mathrm{Im}(Z \oplus \mathbb{Z} \to H_1(K)) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}/\ker(\mathbb{Z} \oplus \mathbb{Z} \to H_1(K)) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}/\mathrm{Im}(\psi) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}/(2, 2) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}_2. \end{aligned}$$

The $* = 0$ case is then just from path-connectivity of K, and the higher groups are 0 from MV. So we are done. ☐

Exercise: What is $H_*(K,\mathbb{Z}_2)$? Clearly something is different if we work in a group where $2 = 0$, since then the ψ map above is $\equiv 0$.

We now go back and prove homotopy invariance of homology and Mayer-Vietoris.

2.2. **Homotopy Invariance of Homology.**

Let C_* and D_* be chain complexes. Let f_* and g_* be chain maps $C_* \to D_*$.

Definition 2.10. *We say that f_* and g_* are* ***chain homotopic*** *if $\exists$ maps $P_n : C_{n-1} \to D_n$ for $n \in \mathbb{Z}$ such that:*

$$dP + Pd = f_* - g_*.$$

Pictorially this means that we have

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & C_{i+2} \arrow[r, "d"] \arrow[d, "f_*"', "g_*"] & C_{i+1} \arrow[r, "d"] \arrow[d, "f_*"', "g_*"] \arrow[dl, red, "P"'] & C_i \arrow[r, "d"] \arrow[d, "f_*"', "g_*"] \arrow[dl, red, "P"'] & C_{i-1} \arrow[r] \arrow[d, "f_*"', "g_*"] \arrow[dl, red, "P"'] & \cdots \\
\cdots \arrow[r, "d"] & D_{i+2} \arrow[r, "d"] & D_{i+1} \arrow[r, "d"] & D_i \arrow[r, "d"] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

and thus the above condition is asking for a certain type of commutativity of the slanted parallelograms

$$\begin{tikzcd}
 & C_i \arrow[r, "d"] \arrow[dl, "P"] & C_{i-1} \arrow[dl, "P"] \\
D_{i+1} \arrow[r, "d"'] & D_i &
\end{tikzcd}.$$

The key point is that chain homotopic maps induce the same maps on homology.

Lemma 2.11. *Suppose f_* and g_* are chain homotopic. Then, $f_* = g_*$ as maps $H(C_*,d) \to H(D_*,d)$.*

Proof. Let $\alpha \in H_n(C_*)$ and suppose $a \in C_n$ is a cycle representing this class. Then we have

$$f_*(a) - g_*(a) = (dP + Pd)(a) = d(Pa)$$

since $da = 0$ as a is a cycle. Hence we see that $f_*(a) - g_*(a)$ is exact, and hence $[f_*(a)] = [g_*(a)] \in H_n(D_*)$. So as $f_*([a]) := [f_*(a)]$, this tells us $f_*(\alpha) = g_*(\alpha)$ and so $f_* = g_*$.

□

Theorem 2.3 (Homotopy Invariance)**.** *Suppose $f, g : X \to Y$ are homotopic. Then $f_* = g_*$ as maps $H_*(X) \to H_*(Y)$.*

Proof. We will show that f_* and $g_* : C_*(X) \to C_*(Y)$ are chain homotopic, and so we are done by Lemma 2.11.

As $f \simeq g$, we know $\exists F : X \times [0,1] \to Y$ such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$. Then let

$$i_0 : X \hookrightarrow X \times [0,1] \quad \text{be} \quad i_0(x) := (x,0)$$

and

$$i_1 : X \hookrightarrow X \times [0,1] \quad \text{be} \quad i_1(x) := (x,1).$$

Then we have $f : F \circ i_0$ and $g = F \circ i_1$, and so $f_* = F_* \circ (i_0)_*$ and $g_* = F_* \circ (i_1)_*$.

So in fact we only need to be able to show that $(i_0)_*$ and $(i_1)_*$ are chain homotopic. We will show this by defining a **prism operator**, which cuts $\Delta^n \times [0,1]$ into $(n+1)$—simplicies.

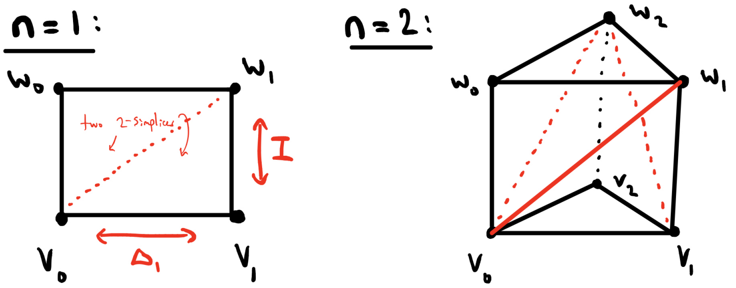


FIGURE 14. An illustration of the prism operator in the cases $n = 1,2$.

In general, consider $\Delta^n \times [0,1] \in \mathbb{R}^{n+1} \times [0,1] \subset \mathbb{R}^{n+2}$. Label the base n—simplex $\Delta^n \times \{0\} = [v_0,\dots,v_n]$ and the top n—simplex $\Delta^n \times \{1\} = [w_0,\dots,w_n]$. Then consider the n—simplicies $[v_0,\dots,v_i, w_{i+1},\dots,w_n]$ and the $(n+1)$—simplicies $[v_0,\dots,v_i,w_i,\dots,w_n]$ (imagine these geometrically, using the above diagrams).

> **Claim 1:** The $(n+1)$—simplicies $[v_0,\dots,v_0,w_i,\dots,w_n]$ exactly fill the prism $\Delta^n \times [0,1]$.
>
> *Proof of Claim 1.* Consider the map $\varphi_i : \Delta^n \to [0,1]$ given by:
>
> $$\varphi_i(t_0,\dots,t_n) = t_{i+1} + \cdots + t_n.$$
>
> Note that all the vertices $[v_0,\dots,v_i,w_{i+1},\dots,w_n]$ lie on the graph graph(φ_i), and so these span an n—simplex inside $\Delta^n \times [0,1]$, which projects homeomorphically to the base. Clearly

$$(\dagger) \qquad \varphi_i \leq \varphi_{i-1} \quad \text{and so} \quad 0 = \varphi_n \leq \varphi_{n-1} \leq \cdots \leq \varphi_{-1} = 1.$$

> The region between graph(φ_i) and graph(φ_{i-1}) is exactly $[v_0,\dots,v_i,w_i,\dots,w_n]$ and this is an $(n+1)$—simplex: the fact that $w_i \notin$ graph(φ_i) shows this set of vertices does satisfy the linear independence condition of being an $(n+1)$—simplex in $\mathbb{R}^{n+2}$.
>
> Then with (†) this shows these $(n+1)$—simplices fill $\Delta^n \times [0,1]$, as required.
>
> ☐

Now define $P : C_n(X) \to C_{n+1}(X \times [0,1])$ by

$$\sigma \longmapsto \sum_i (-1)^i (\sigma \times \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}.$$

Claim 2: $dP + Pd = (i_1)_* - (i_0)_*$.

[Geometrically this says that the boundary of the prism $\Delta^n \times [0,1]$ is the disjoint union of the prism on the boundary, the top, and the base.]

Proof of Claim 2. We have (from the definition of P and d, splitting up the cases where d throws out a v_i or w_j index):

$$\begin{aligned} dP(\sigma) = \sum_{j\leq i} &(-1)^i(-1)^j(\sigma \times \mathrm{id})|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]} \\ &+ \sum_{j\geq i}(-1)^i(-1)^{j+1}(\sigma\times \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]} \\ = &\underbrace{(\sigma\times\mathrm{id})|_{\hat{v}_0,w_0,\ldots,w_n]}}_{j=i=0 \text{ in 1st sum}} - \underbrace{(\sigma\times\mathrm{id})|_{[v_0,\ldots,v_n,\hat{w}_n]}}_{j=i=n \text{ in 2nd sum}} \\ &+ \underbrace{\sum_{j<i}(\cdots) + \sum_{j>i}(\cdots)}_{\text{turns out to be } -P(d\sigma) \text{ - } \textbf{Exercise} \text{ to check}}. \end{aligned}$$

The first term here is the top of the prism, i.e. $(i_1)_*\sigma$. The second term is the base, i.e. $(i_0)_*\sigma$. So thus we get

$$d(P\sigma) = (i_1)_*\sigma - (i_0)_*\sigma - P(d\sigma)$$

and so we are done.

□

With this the proof is completed.

□

Remark: For cochain maps of cochain complexes C^* and D^*, we say that $f^*, g^* : D^* \to C^*$ are **cochain homotopic** if $\exists$ maps $P^* : C^{i+1} \to D^i$, $i \in \mathbb{Z}$, such that

$$dP^* + P^*d = f^* - g^*.$$

In this scenario, the usual prism operator $P : C_n(X) \to C_{n+1}(X \times [0,1])$ dualises to:

$$P^* : \underbrace{\mathrm{Hom}(C_{n+1}(X\times[0,1]),\mathbb{Z})}_{\cong C^{n+1}(X\times[0,1])} \to \underbrace{\mathrm{Hom}(C_n(X),\mathbb{Z})}_{\cong C^n(X)}$$

and taking duals, we have

$$dP + Pd = f_* - g_* \quad \Longrightarrow \quad dP^* + P^*d = f^* - g^*.$$

Given these observations, one sees that singular cohomology is also homotopy invariant, i.e.

$$\text{if } f \simeq g, \quad \text{then} \quad f^* = g^* \quad \text{as maps} \quad H^*(Y) \to H^*(X).$$

Remark: Proofs for cohomology will tend to just be as above: once you have the result for homology, you just dualise everything and observe that the same proof works. We will stop spelling out the details for cohomology explicitly unless there is something crucially different.

2.3. **Mayer-Vietoris (MV).**

Before proving the Mayer-Vietoris (MV) property, we need a bit more algebra. Recall that a chain complex was said to be **exact** if it has trivial homology, i.e. $H_*(C_*, d) = 0$, or equivalently if $\ker(d_n) = \mathrm{Im}(d_{n+1})$ for all $n \in \mathbb{Z}$.

Definition 2.11. *A* ***short exact sequence (s.e.s)*** *is an exact sequence of the form*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Lemma 2.12. *In a s.e.s, α is injective, β is surjective, and β induces an isomorphism $B/A \overset{\cong}{\longrightarrow} C$.*

Proof. Exercise. □

Definition 2.12. *A* ***s.e.s of chain complexes*** $0 \to A_* \to B_* \to C_* \to 0$ *is a diagram*

$$\begin{tikzcd}
 & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\alpha"] \arrow[d, "d"] & B_{n+1} \arrow[r, "\beta"] \arrow[d, "d"] & C_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\alpha"] \arrow[d, "d"] & B_n \arrow[r, "\beta"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha"] \arrow[d, "d"] & B_{n-1} \arrow[r, "\alpha"] \arrow[d, "d"] & C_{n-1} \arrow[r] \arrow[d, "d"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

such that all squares commute, the columns are chain complexes (i.e. $d^2 = 0$) and the rows are exact (i.e. $\ker(\beta) = Im(\alpha)$ *for all n).*

Proposition 2.1. *Given a s.e.s of chain complexes $0 \to A_* \to B_* \to C_* \to 0$, $\exists$ an associated l.e.s (long exact sequence) in homology:*

$$\cdots \longrightarrow H_i(A_*) \xrightarrow{\alpha} H_i(B_*) \xrightarrow{\beta} H_i(C_*) \xrightarrow{\partial} H_{i-1}(A_*) \longrightarrow H_{i-1}(B_*) \longrightarrow \cdots$$

i.e. $\exists$ such maps ∂ such that this sequence is exact.

Note: This is similar to MV: ∂ is a 'boundary map', which lowers the degree.

Proof. The technique for this proof is so-called "diagram chasing".

We shall construct ∂ and leave verification of exactness for the 1st example sheet.

Let $\gamma \in H_n(C_*)$ be represented by a cycle $c_n \in C_n$, which is a cycle in the C_*-chain complex. From the s.e.s of chain complexes we have the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\alpha"] \arrow[d, "d"] & B_n \arrow[r, "\beta"] \arrow[d, "d"] \arrow[d, bend right=60, "b_n"'] & C_n \arrow[r] \arrow[d, "d"] \arrow[l, bend right=50, "c_n"'] & 0 \\
0 \arrow[r] & \underset{a_{n-1}}{A_{n-1}} \arrow[r, "\alpha"] & B_{n-1} \arrow[r, "\beta"] \arrow[l, bend left=50, "db_n"] & C_n \arrow[r] & 0.
\end{tikzcd}$$

By exactness we know thar β is surjective, and so $\exists b_n \in B_n$ such that $\beta(b_n) = c_n$. But then we have

$$\beta(d(b_n)) = d(\beta(b_n)) = d(c_n) = 0$$

since c_n is a cycle, and so $db_n \in \ker(\beta) = \operatorname{Im}(\alpha)$. So $\exists a_{n-1} \in A_{n-1}$ such that $\alpha(a_{n-1}) = db_n$.

Now, $\alpha(da_{n-1}) = d(\alpha(a_{n-1})) = d(db_n) = 0$ since $d^2 = 0$. So as α is injective, this tells us that $da_{n-1} = 0$, and so a_{n-1} defines a homology class $[a_{n-1}]$.

Then we define:

$$\partial[c_n] := [a_{n-1}] \qquad (= [\alpha^{-1}(d\beta^{-1}(c_n))]).$$

Note that we made a choice of b_n in the first step of the above. If we change that to b_n', say, then a_{n-1} changes to $a_{n-1} + da_n$, where $b_n' = b_n + \alpha(a_n)$. So as $[a_{n-1} + da_n] = [a_{n-1}]$, this shows that $\partial[c_n]$ is independent of the choice of b_n.

So it remains to check:

- ∂ is independent of the choice of cycle c_n representing $[c_n]$.
- The resulting map $\partial : H_n(C_*) \to H_{n-1}(A_*)$ is a homomorphism.
- The resulting l.e.s is exact (need to check this in 6 places of the diagram).

These are left as **exercises** to check. Then the proof is complete.

□

We can use Proposition 2.1 to generate canonical homologies.

Example 2.2 (Relative Homology)**.** *Let X be a topological space and $A \subset X$ a $\overline{\text{subspace}}$. Then we have $C_n(A) \subset C_n(X)$ is a subspace, and moreover $C_*(A) \subset C_*(X)$ is preserved by d. So there is an induced quotient complex*

$$C_n(X,A) := \frac{C_n(X)}{C_n(A)}$$

with the inherited differential.

Then by construction we have that

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X,A) \longrightarrow 0$$

is a s.e.s of chain complexes. So applying Proposition 2.1, we get an associated l.e.s

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X,A) \longrightarrow H_{i-1}(A) \longrightarrow \cdots$$

called the ***l.e.s of the pair*** (X,A)*. This gives rise to the* ***relative homology*** *groups, $H_i(X,A)$.*

Notation: *We write $H_*(X,A) := H_*(C_*(X,A), d)$.*

Note: How should we think about relative homology? A cycle in relative homology is a chain in X whose boundary lies in A.

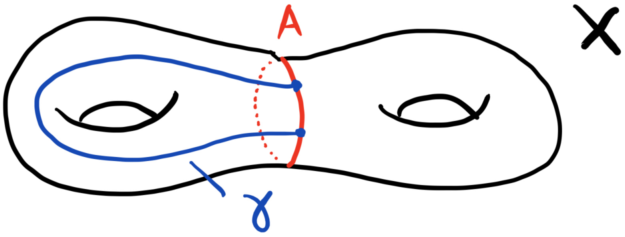


FIGURE 15. An illustration of what can happen with relative homology. Here γ is a cycle in $C(X,A)$ despite not being one in $C_1(X)$, since relative homology kills off these curves with endpoints in A.

Note: $H_*(X,A)$ is *natural* for maps of pairs. By a map of pairs $f : (X,A) \to (Y,B)$ we mean a map $f : X \to Y$, with $f(A) \subset B$ for $A \subset X$, $B \subset Y$. By naturality, we mean that if we have such a map of pairs then f induces a map $f_* : H_*(X,A) \to H_*(Y,B)$ in the natural way.

Example 2.3 (Bockstein Homomorphisms)**.** *An exact sequence of abelian groups*

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

induces maps on chain complexes, and we get a s.e.s of chain complexes

$$0 \to C_*(X,G_1) \to C_*(X,G_2) \to C_*(X,G_3) \to 0.$$

E.g. the s.e.s's

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{\times m} \mathbb{Z}_{m^2} \longrightarrow \mathbb{Z}_m \longrightarrow 0$$

yield boundary maps (for the l.e.s in homology) $H_i(X,\mathbb{Z}_m) \to H_{i-1}(X,\mathbb{Z})$ and $H_i(X,\mathbb{Z}_m) \to H_{i-1}(X,\mathbb{Z}_m)$ respectively, which are homomorphisms. Such maps are known as **Bockstein homomoprhisms**.

Example 2.4 (Mayer-Vietoris)**.** *Let $U = \{U_\alpha : \alpha \in A\}$ be an open cover of X, or more generally a cover of X by subspaces whose interiors cover. Then let:*

$$C_n(X,U) := \left\{ \sum_{i=1}^{N} h_i\sigma_i : N \in \mathbb{N},\ h_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \text{ has image } \mathrm{Im}(\sigma_i) \subset U_{\alpha(i)} \text{ for some } \alpha(i) \in A \right\}.$$

So this is the subgroup of $C_*(X)$ comprising of chains, each of whose constituent simplicies lie wholly in some set belonging to U. This is a ***subcomplex*** *of $C_*(X)$.*

To prove Mayer-Vietoris, we shall use Example 2.4 as well as the *small simplicies theorem*.

Proposition 2.2 (The Small Simplicies Theorem)**.** *The inclusion $C_*(X,U) \hookrightarrow C_*(X)$ of chain complexes induces an isomorphism*

$$H(C_*(X,U)) \xrightarrow{\cong} H(C_*(X)) =: H_*(X)$$

Proof. Momentarily. □

Given this result, let us quickly see how it proves Mayer-Vietoris. Let $U = \{A,B\}$ be a cover of X by two (open) sets. Then these is an obvious [**Exercise** to check] s.e.s of chain complexes:

$$o \longrightarrow C_*(A\cap B) \xrightarrow{\varphi} C_*(A)\oplus C_*(B) \xrightarrow{\psi} C_*(X,U) \longrightarrow 0$$

where $\varphi(\sigma) = (\sigma,\sigma)$ and $\psi(u,v) = u - v$. [Surjectivity on the RHS uses that we only consider $C_*(X,U) \subset C_*(X)$.]

Hence this induces a l.e.s on homology by Proposition 2.1, which is the Mayer-Vietoris sequence, since $H_*(C_*(X,U)) \cong H_*(X)$, by the small simplicies theorem.

Remark: The construction of the map $\partial : H_*(C_*(X,U)) \to H_{*-1}(A\cap B)$ exactly fits our description of the MV boundary map ∂_{MV}.

Remark: If U is a cover of X and V is a cover of Y, and $f : X \to Y$ takes each set in U wholly into some set of V, then the map $f_* : C_*(X) \to C_*(Y)$ preserves the subcomplexes, i.e.

$$f_*(C_*(X,U)) \subset C_*(Y,V).$$

This then gives [**Exercise** to check] the naturality of the MV sequence under maps of pairs, mentioned before.

Thus once we have proven the small simplicies theorem we will have proven MV and the associated comments/addenda.

To set up for the proof, we shall first construct a barycentric subdivision operator, which will be chain homotopic to the identity via a prism operator.

If $\sigma = [v_0, \ldots, v_n]$ is a simplex, then set

$$b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

called the **barycentre** of the simplex (just the centre of mass of the vertices). Write $b_n \in \mathbb{R}^{n+1}$ for the barycentre of the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$.

We will construct a subdivision operator

$$\begin{tikzcd} C_*(X) \arrow[r, "\varphi"] & C_*(X,U) \arrow[l, bend right=60, dashed, "\text{inclusion}"'] \end{tikzcd}$$

which is a chain map such that $\exists D : C_*(X) \to C_{*+1}(X)$ with

$$dD + Dd = 1 - (\text{inclusion} \circ \varphi).$$

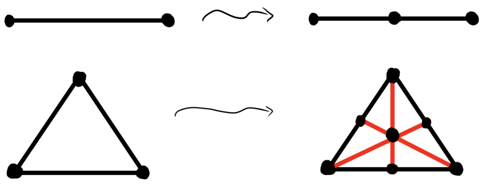


FIGURE 16. An illustration of how we want the barycentric subdivision operator to work. The top image shows the subdivision of a 1-simplex, whilst the lower image shows the subdivision of a 2-simplex. Note we generate new 2-simplicies, 1-simplicies, and 0-simplicies.

i.e. given a n-simplex, to generate new subdivided n-simplicies we subdivide the boundary and then 'cone off' the result to the barycentre.

More formally, let $\sigma : \Delta^i \to \Delta^n$ be an i–simplex in Δ^n, i.e. a generator of $C_i(\Delta^n)$. Then we define the **coning operator** by

$$\text{Cone}_i^{\Delta^n}(\sigma) : \Delta^{i+1} \to \Delta^n \quad \text{sending} \quad (t_0, \ldots, t_{i+1}) \mapsto t_0 b_n + (1-t_0)\sigma\left(\frac{(t_1, \ldots, t_{i+1})}{1-t_0}\right).$$

which is just a convex combination of the barycentre and the simplex.

Extending this operator linearly, we obtain a map

$$\text{Cone}_i^{\Delta^n} : C_i(\Delta^n) \to C_{i+1}(\Delta^n)$$

and so the describes how we obtain $(i+1)$–simplicies from i–simplicies.

One can then check [**Exercise**] that

$$d\left(\mathrm{Cone}_i^{\Delta^n}(\sigma)\right) = \begin{cases} \sigma - \mathrm{Cone}_{i-1}^{\Delta^n}(d\sigma) & \text{if } i > 0 \\ \sigma - \varepsilon(\sigma) b_n & \text{otherwise} \end{cases}$$

where $\varepsilon : C_0(\Delta^n) \to \mathbb{Z}$ sending $\sum_i n_i p_i \longmapsto \sum_i n_i$.

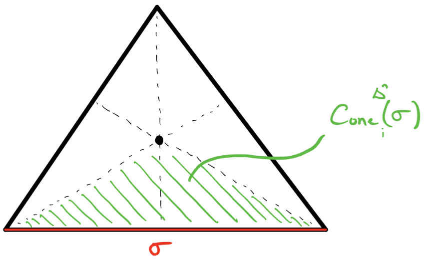


FIGURE 17. An illustration of the coning map. We take a simplex and form the 'cone' with the barycentre, i.e. we 'cone off' to the barycentre.

Thus if we define $c_* : C_*(\Delta^n) \to C_*(\Delta^n)$ by:

$$c_*(\sigma) := \begin{cases} \varepsilon(\sigma) b_n & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

then with the above we see that

$$d\left(\mathrm{Cone}^{\Delta^n}\right) + \mathrm{Cone}_*(\Delta^n) \circ d = \mathrm{id}_{C_*(\Delta^n)} - c_*$$

which is looking good for a chain homotopy.

We now want to define the full barycentric subdivision operator on general simplicies X and not just Δ^n, i.e. construct $\varphi_n^X : C_n(X) \to C_n(X)$. We do this inductively. First we set $\varphi_0^X = \mathrm{id}_{C_0(X)}$. Then for $n > 0$ we define:

$$\varphi_n^X : \sigma \longmapsto \sigma_*\left(\mathrm{Cone}_{n-1}^{\Delta^n}\left(\varphi_{n-1}^{\Delta^n}(d\iota_n)\right)\right)$$

where $\iota_n : \Delta^n \xrightarrow{\mathrm{id}} \Delta^n$ (so $\iota_n \in C_n(\Delta^n)$) and $\sigma_* : C_n(\Delta^n) \to C_n(X)$. Thus simply says that we subdivide the boundaries cone them off in Δ^n just as above, and then map the results under σ into X.

Definition 2.13. *We say that a collection of chain maps $(\varphi_X)_X$ (one for every space X), $\varphi^X : C_*(X) \to C_*(X)$, are* ***natural*** *if whenever $f : X \to Y$ we have:*

$$f_* \circ \varphi^X = \varphi^Y \circ f_*.$$

In particular naturality says that:

$$\varphi_n^X(\sigma) = \varphi_n^X(\sigma_*(\iota_n)) = \sigma_*(\varphi_n^{\Delta^n}(\iota_n)).$$

Similarly we have a notion of a **natural family of chain homotopies**, $P^X : C_*(X) \to C_{*+1}(X)$.

Continuing with our construction of the subdivision operator, define inductively **prism maps** P_n via:

$$P_n : \sigma \longmapsto \sigma_*\left(\mathrm{Cone}_n^{\Delta^n}\left(\varphi_n^{\Delta^n}(\iota_n) - \iota_n - P_{n-1}^{\Delta^n}(d\iota_n)\right)\right).$$

Geometrically, we take $\Delta^n \times [0,1]$, subdivide the top $\Delta^n \times \{1\}$ and boundary, and join $\Delta^n \times \{0\}$ and $\partial\Delta^n \times \{1\}$ to $b_n \in \Delta^n \times \{1\}$. [See Hatcher's book for pictures.]

The key lemma is then:

Lemma 2.13. $\varphi^X : C_*(X) \to C_*(X)$ *is a natural chain map.*

Moreover, $P^X : C_*(X) \to C_{*+1}(X)$ *is a natural chain homotopy from* φ^X *to the identity, i.e.*

$$dP_n^X + P_{n-1}^X \circ d = \varphi_n^X - \mathrm{id}_{C_n(X)} \qquad \textit{for all spaces } X \textit{ and all } n.$$

Proof. Omitted (it doesn't add to understanding to check the details live in a lecture - all the details to check this have been provided though) [**Exercise** to prove].

□

To prove the small simplicies theorem we need two more results about what happens in the subdivision.

Lemma 2.14. *Let* $[v_0, \ldots, v_n] \subset \mathbb{R}^N$ *be a simplex with Euclidean diameter (defined in the normal way)* $\mathrm{diam}([v_0, \ldots, v_n])$*. Then each simplex of* $\varphi_n^{\Delta^n}([v_0, \ldots, v_n])$*, i.e. of its barycentric subdivision, has diameter* $\leq \frac{n}{n+1} \cdot \mathrm{diam}([v_0, \ldots, v_n])$*.*

Proof. **Exercise**. □

So the above tells us that if we barycentric subdivide, all resulting simplicies get smaller in diameter and we have this fixed bound on how much. So if we were to repeatedly subdivide, we could ensure that all simplicies were eventually so small that they lie a given set of our cover U. This is exactly what the following result says.

Corollary 2.5. *We have*

- *(i) If* $\sigma \in C_n(X, U)$*, then* $\varphi_n^X(\sigma) \in C_n(X, U)$*.*
- *(ii) If* $\sigma \in C_n(X)$*, then* $\exists k \gg 0$ *such that* $(\varphi_n^X)^k(\sigma) \in C_n(X, U)$*.*

Proof. (i): Obvious - if the image of the simplex already lies in some U_i in the cover of U, then since subdividing can only make the simplex smaller (without shifting it), the subdivided σ must still lie in U_i.

(ii): If $\sigma \in C_n(X)$, then σ is a finite sum of $n-$simplicies and so it suffices to consider the case $\sigma : \Delta^n \to X$.

Then if $U = \{U_\alpha : \alpha \in A\}$, then as $X \subset \bigcup_\alpha U_\alpha$ we have that $\{\sigma^{-1}(U_\alpha) : \alpha \in A\}$ is an open cover of Δ^n, which is a compact metric space.

So hence this open cover has a **Lebesgue number**, i.e. $\exists \varepsilon > 0$ such that any ε—ball in Δ^n is contained in some set of the cover. But then as $\left(\frac{n}{n+1}\right)^k \to 0$ as $k \to \infty$, by Lemma 2.14 we can choose $k \gg 0$ such that each simplex in the barycentric subdivision has diameter $< \varepsilon$, and so lies in some $\sigma^{-1}(U_\alpha)$ of the cover, which then implies that the simplex in X lies in some U_α, i.e. $(\varphi_n^X)^k(\sigma) \in C_n(X,U)$. [Here $(\varphi_n^X)^k$ means φ_n^X composed with itself k times.]

□

Now we can prove the small simplicies theorem.

Proof of Small Simplicies Theorem. Let $\mathscr{U} : H_*(X,U) \to H_*(X)$ be the natural map coming from the inclusion $C_*(X,U) \hookrightarrow C_*(X)$ as a subcomplex. We will show that this is a bijection.

Now let $[c] \in H_n(X)$. By Corollary 2.5, $\exists k \gg 0$ such that $(\varphi_n^X)(c) \in C_n(X,U)$. Now as φ^X is chain homotopic to the identity (via the prism operator - see Lemma 2.13), and since chain homotopy equivalence is an equivalence relation [**Exercise** to check], we see that $(\varphi^X)^k$ is chain homotopic to the identity.(i)

So hence $\exists F^k$ such that:

$$dF^k + F^k d = (\varphi^X)^k - \mathrm{id}.$$

So evaluating this at c we get (since $dc = 0$)

$$(\varphi^X)^k(c) = c + d(F^k(c)) \quad \Longrightarrow \quad [(\varphi^X)^k(c)] = [c]$$

i.e. c is homologous to $(\varphi^X)^k$, and so hence $[c]$ lies in the image of $\mathscr{U}$ since $(\varphi_n^X)(c) \in C_n(X,U)$.

Hence this shows that $\mathscr{U}$ is surjective. So all that remains is to show that $\mathscr{U}$ is injective.

Suppose that $[c] \in H_n(X,U)$ and $U([c]) = 0$. Then $\exists z \in C_{n+1}(X)$ such that $dz = c$. Again by Corollary 2.5, $\exists k \gg 0$ such that $(\varphi^X)^k(z) \in C_{n+1}(X,U)$, and so

$$(\varphi_{n+1}^X)^k(z) - z = d(F^k(z)) + F^k(dz) \quad \Longrightarrow \quad \underbrace{d\left((\varphi_{n+1}^X)^k(z)\right)}_{\in C_{n+1}(X,U)} - \underbrace{d\left(F^k(dz)\right)}_{\in C_{n+1}(X,U)} = dz = c$$

i.e. $c \in C_{n+1}(X,U)$, where we have used the naturality of the prism operator P and F^k.

So hence this shows $[c] = 0$ in $H_n(X,U)$ and so $\mathscr{U}$ is injective and thus is an isomorphism. So done.

□

So we have now honestly proven everything we have claimed up to this point. Our next goal is to prove *excision*, which will help us calculate relative homology groups.

(i) i.e. $\varphi^X \circ \varphi^X \simeq \varphi^X \circ \mathrm{id} = \varphi^X \simeq \mathrm{id}$.

3. EXCISION

Homology groups of pairs/relative homology carry more information than homology itself, since $H_*(X,\emptyset)=H_*(X)$. More fundamental than the Mayer-Vietoris theorem is the idea of *excision*, which is all about when removing subsets does not change the relative homology. Before the statement of the result, we need another algebraic lemma.

Lemma 3.1 (The 5-Lemma). *Suppose we have a commuting diagram of abelian groups*

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\alpha"] & B \arrow[r, "d"] \arrow[d, "\beta"] & C \arrow[r, "d"] \arrow[d, "\gamma"] & D \arrow[r, "d"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "d'"] & B' \arrow[r, "d'"] & C' \arrow[r, "d'"] & D' \arrow[r, "d'"] & E'
\end{tikzcd}$$

such that the rows are exact. Then if $\alpha,\beta,\delta,\varepsilon$ are isomorphisms, so is γ.

Proof. **Exercise** in diagram chasing. We will show here that γ is injective and leave the rest as an exercise.

So let $c\in C$ have $\gamma(c)=0$. Then by commutativity we have

$$0=d'(\gamma(c))=\delta(dc)$$

and so as δ is an isomorphism it is injective and so $dc=0$, i.e. $c\in\ker(d)=\operatorname{Im}(d:B\to C)$ by exactness. So $\exists b\in B$ such that $db=c$.

By then again by commutativity we have

$$d'(\beta(b))=\gamma(db)=\gamma(c)=0$$

and so $\beta(b)\in\ker(d')=\operatorname{Im}(d':A'\to B')$. So hence $\exists a'\in A'$ with $\beta(b)=d'a'$. But then since α is an isomorphism (and so surjective), $\exists a\in A$ with $\alpha(a)=a'$. But then by commutativity,

$$\beta(da)=d'(\alpha(a))=d'(a')=\beta(b)$$

and thus as β is an isomorphism we must have $da=b$. But then

$$0=d^2a=db=c$$

i.e. $c=0$ and so γ is injective. Surjectivity is then very similar.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] \arrow[d, bend right, "\beta(b)"'] & \overset{c}{C} \arrow[r] \arrow[d] \arrow[l, bend right, "b"'] & \overset{dc=0}{D} \arrow[r] \arrow[d, "\delta"] & E \arrow[d] \\
A' \arrow[r] \arrow[u, bend left, "a"] & B' \arrow[r] \arrow[l, bend left, "a'"] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

An illustration of the diagram chase for injectivity in the 5-Lemma.

□

The fundamental result is then the following.

Theorem 3.1 (The Excision Theorem). *Let X be a space and $Z,A \subset X$ with $\overline{Z} \subset Int(A)$. Then the inclusion of pairs $(X,Z) \hookrightarrow (X,A)$ induces an isomorphism:*

$$H_*(X\backslash Z, A\backslash Z) \xrightarrow{\cong} H_*(X,A)$$

and similarly for cohomology.

Proof. Let $B = X\backslash Z$. Then $X = A \cup B$ is a covering by sets whose interiors cover. We then have two s.e.s's of chain complexes (here $U = \{A,B\}$):

$$\begin{tikzcd}
0 \arrow[r] & C_*(A) \arrow[r] \arrow[d] & C_*(X,U) \arrow[r] \arrow[d] & \frac{C_*(X,U)}{C_*(A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & \frac{C_*(X)}{C_*(A)} \arrow[r] & 0
\end{tikzcd}$$

where we note that $C_*(X)/C_*(A) =: C_*(X,A)$. Thus we get l.e.s's in homology, with natural maps between them:

$$\begin{tikzcd}
H_i(A) \arrow[r] \arrow[d, "\cong"] & H_i(X,U) \arrow[r] \arrow[d, "\cong"] & H_i\left(\frac{C_*(X,U)}{C_*(A)}\right) \arrow[r] \arrow[d] & H_{i-1}(A) \arrow[r] \arrow[d, "\cong"] & H_{i-1}(X,U) \arrow[r] \arrow[d, "\cong"] & \cdots \\
H_i(A) \arrow[r] & H_i(X) \arrow[r] & H_i(X,A) \arrow[r] & H_{i-1}(A) \arrow[r] & H_{i-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where the maps $H_*(X,U) \xrightarrow{\cong} H_*(X)$ are isomorphisms from the small simplicies theorem. Thus the 5-Lemma gives that the middle map must be an isomorphism as well, i.e. the map

$$\frac{C_*(B)}{C_*(A\cap B)} \equiv \frac{C_*(X,U)}{C_*(A)} \to \frac{C_*(X)}{C_*(A)}$$

induces an isomorphism on homology. So in fact the inclusion induces isomorphisms

$$H_*(B, A\cap B) \xrightarrow{\cong} H_*(X,A).$$

Noting that $H_*(B,A\cap B) \equiv H_*(X\backslash Z, A\backslash Z)$, we are done.

□

Exercise: Suppose that $A \subset X$ has an open neighbourhood $A \subset V \subset X$ such that the inclusion $A \hookrightarrow V$ is a homotopy equivalence, then (show that)

$$H_*(X,A) \xrightarrow{\cong} H_*(X,V)$$

via the natural map of pairs $(X,A) \hookrightarrow (X,V)$. [*Hint:* This is an exercise in using the 5-Lemma.]

This is just saying that if nothing interesting happens between A and V which distinguishes them, then we may as well work with the larger set V/smaller set A as appropriate.

Definition 3.1. *A pair (X,A) is a* **good pair** *if $\exists$ an open neighbourhood $A \subset V \subset X$ of A in X such that:*

(i) $\overline{A} \subset V$ (although we will assume A is closed - see below)

(ii) The inclusion $A \hookrightarrow V$ is a deformation retract$^{(\mathrm{ii})}$.

Note: It is safer to assume that A itself is closed in the definition of a good pair, since we want the map $X\backslash A \to (X/A)\backslash(A/A)$ to be a quotient map. However if X (and hence U) is Hausdorff, our original definition actually forces A to be closed.

The key point about good pairs is that there is not special information in them, and so we can shrink A to a point without changing relative homology (i.e. cycles in A are already boundaries). The following makes this precise:

Proposition 3.1. *Suppose (X,A) is a good pair. Then the natural map*

$$H_*(X,A) \longrightarrow H_*(X/A,A/A) \quad \equiv \quad H_*(X/A,\{point\}) \quad \equiv \quad \tilde{H}_*(X/A)$$

is an isomorphism.

Proof. Momentarily. ☐

Remark: For any space X, its **reduced homology** is defined by:

$$\tilde{H}_*(X) := H_*(X,\{\text{point}\}).$$

It is often defined as the homology of the augmented chain complex:

$$\cdots \longrightarrow C_i(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

i.e. the usual chain complex with the addition of $\varepsilon : C_0(X) \to \mathbb{Z}$, which is defined by

$$\varepsilon : \sum_i n_i p_i \longmapsto \sum_i n_i.$$

Concretely this tells us:

$$H_*(X) \cong \begin{cases} \tilde{H}_*(X) & \text{if } * > 0 \\ \tilde{H}_*(X) \oplus \mathbb{Z} & \text{if } * = 0 \end{cases}$$

although the isomorphism is non-canonical.

Example 3.1 (Important Class of Good Pairs)**.** *Let X be a smooth manifold and let $A \subset X$ be a closed smooth submanifold (closed $\equiv$ compact without boundary). Then the tubular neighbourhood theorem $\Rightarrow$ (X,A) is a good pair.*

[(ii)] i.e. $\exists H : V \times [0,1] \to V$ such that $H|_{V\times\{0\}} = \mathrm{id}_V$, $H|_{V\times\{1\}}$ has image A, and $H|_{A\times\{t\}} = \mathrm{id}_A$ for all $t \in [0,1]$, i.e. we can shrink the neighbourhood down to A whilst never moving A.

Proof of Proposition 3.1. Homotopy invariance says $H_*(A) \xrightarrow{\cong} H_*(U)$, and then the 5-Lemma gives $H_*(X,A) \xrightarrow{\cong} H_*(X,U)$ [here $A \hookrightarrow U$ is a deformation retract as in the definition of a good pair, so $A \subset \overline{A} \subset U \subset X$ with U open in X].

Then since $A \hookrightarrow U$ is a deformation retract, it induces a deformation retract $\{\text{point}\} = A/A \hookrightarrow U/A$, and so we also get an isomorphism

$$H_*(X/A, A/A) \xrightarrow{\cong} H_*(X/A, U/A).$$

Then we have:

$$\begin{tikzcd}
H_*(X,A) \arrow[r, "\cong"] \arrow[d] & H_*(X,U) & H_*(X\backslash A, U\backslash A) \arrow[l, "\cong", "\text{Excision}"'] \arrow[d, "\cong"] \\
H_*(X/A, A/A) \arrow[r, "\cong"] & H_*(X/A, U/A) & H_*\left(\frac{X}{A}\backslash\frac{A}{A}, \frac{U}{A}\backslash\frac{A}{A}\right) \arrow[l, "\cong", "\text{Excision}"']
\end{tikzcd}$$

where the red arrow is an isomorphism since the projection $X \to X/A$ induces a homomorphism of pairs, $(X\backslash A, U\backslash A) \cong \left(\frac{X}{A}\backslash\frac{A}{A}, \frac{U}{A}\backslash\frac{A}{A}\right)$.

So following this diagram around we see that the map $H_*(X/A, A/A) \to H_*(X,A)$ is an isomorphism and so we are done.

□

Lemma 3.2. *Let M be a connected manifold, and $x \in M$. Then:*

$$H_*(M, M\backslash\{x\}) \cong \begin{cases} \mathbb{Z} & \text{if } * = \dim_{\mathbb{R}}(M) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition of being a manifold, we know $\exists$ an open neighbourhood $U \subset M$ of x in M with $U \cong \mathbb{R}^n$. Via excising $M\backslash U$, excision then says

$$H_*(M, M\backslash\{x\}) \cong H_*(U, U\backslash\{x\}).$$

Now as $U \cong \mathbb{R}^n$, we have

$$H_*(U, U\backslash\{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$$

and we know

$$H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \xrightarrow{\cong} H_{*-1}(\mathbb{R}^n\backslash\{0\})$$

from the l.e.s of a pair, for $* > 1$. Thus we are done since $\mathbb{R}^n\backslash\{0\} \simeq S^{n-1}$.

□

Remark: Later we will define *orientations* of manifolds as coherent classes of generators for these groups $H_n(M, M\backslash\{x\})$.

We now want to understand how to compute degrees of maps of spheres, as this will be used when working with cellular homology.

4. CELL COMPLEXES AND CELLULAR HOMOLOGY

4.1. Degree Revisited.

Let $f: S^n \to S^n$ be a map. Assume that $\exists y \in S^n$ with $f^{-1}(y) = \{x_1, \ldots, x_n\}$ is finite. Then $\exists$ a disjoint collection of open discs U_i with $x_i \in U_i$ for each i, and an open disc $V \ni y$ such that $f|_{U_i}: U_i \to V$.

Definition 4.1. *The **local degree** of f at x_i, denoted $\deg_{x_i}(f)$, is the degree of the induced map*

$$\begin{tikzcd} H_n(U_i, U_i\backslash\{x_i\}) \arrow[r] \arrow[d, "\cong"] & H_n(V, V\backslash\{y\}) \arrow[d, "\cong"] \\ \mathbb{Z} & \mathbb{Z} \end{tikzcd}.$$

Note: Both of the groups in the above are identified with $H_n(S^n, S^n\backslash\{\text{point}\} \cong H_n(S^n)$ (with this isomorphism coming from the l.e.s). So hence $\deg_{x_i}(f)$ is well-defined, and not just up to sign.

Proposition 4.1 (Local Degree Formula)**.** *In the above setting, we have*

$$\deg(f) = \sum_{i=1}^{k} \deg_{x_i}(f)$$

i.e. this global invariant can be found via a sum of local behaviours.

Proof. We have (where black arrows are the maps we have, whilst the red arrows are other things we know we can include between the black arrows to help us):

$$\begin{tikzcd} & H_n(U_i, U_i\backslash\{x_i\}) \arrow[r, "\times \deg_{x_i}(f)"] \arrow[d, "i"] \arrow[dl, bend right, "excision"', "\cong"] & H_n(V, V\backslash\{y\}) \arrow[d, "excision"', "\cong"] \\ H_n(S^n, S^n\backslash\{x_i\}) & H_n(S^n, S^n\backslash\{x_1, \ldots, x_n\}) \arrow[l] \arrow[r, "\hat{f}_*"] & H_n(S^n, S^n\backslash\{y\}) \arrow[d, "l.e.s"', "\cong"] \\ & H_n(S^n) \arrow[u, "\varphi"] \arrow[r, "\times \deg(f)"] \arrow[ul, bend left, "l.e.s"] & H_n(S^n) \end{tikzcd}$$

Thus we have

$$H_n(S^n, S^n\backslash\{x_1, \ldots, x_n\}) \underset{\text{excision}}{\cong} H_n(\amalg_i U_i, \amalg(U_i\backslash\{x_i\})) = \bigoplus_{i=1}^{k} H_n(U_i, U_i\backslash\{x_i\})$$

so I is the inclusion of each summand. Also we know $\varphi(1) = (1, \dots, 1)$ since the bottom left square commutes. But then since the bottom right part of the diagram commutes, we have

$$\deg(f) = f_*(1) = \sum_{i=1}^{k} \hat{f}_*(0, \dots \underbrace{, 1,}_{i\text{'th place}} \dots, 0)$$

and then this equals $\sum_{i=1}^{k} \deg_{x_i}(f)$, by commutativity of the top right square of the diagram. So done.

□

Remark: If M, N are smooth compact n-manifolds and $f : M \to N$ is smooth, then Sard's theorem says that for almost all $y \in N$ (and so in particular a dense set of y), $f^{-1}(y)$ is indeed finite.

Example 4.1. *Let $p(z)$ be a complex polynomial. Then p extends to a continuous map $\hat{p} : \mathbb{C} \cup \{\infty\} = S^2 \to S^2 = \mathbb{C} \cup \{\infty\}$, with $\deg(\hat{p}) = \deg(p)$.*

Remark: *The formula $\deg(f) = \sum_{i=1}^{k} \deg_{x_i}(f)$ in particular says that f is surjective if $\deg(f) \neq 0$. So the local degree formula, coupled with the above, can be thought of as a generalisation of the fundamental theorem of algebra.*

Proof. (Sketch) If $p(z) = z^k + a_1 z^{k-1} + \cdots + a_{k-1} z + a_k$, then $\hat{p} \simeq \varphi$ on S^k, where φ is the map $z \mapsto z^k$ [recall ideas from the start of the course].

Then we know $\varphi^{-1}(1) = \{\xi_1, \dots, \xi_k\}$ are the roots of unity, and near each ξ_i, φ is a local homomorphism, and those homomorphisms differ by a rotation of S^2. Indeed near each ξ_i, φ is well approximated by $\mathrm{d}\varphi|_{\xi_i}$, which is a $\mathbb{C}$-linear map. Recall that if we showed that if $A \in GL_n(\mathbb{R})$, then A acts on $H_{n-1}(S^{n-1}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$ by $\det(A)$.

Now a $\mathbb{C}$-linear map has determinant > 0 when viewed as an element of $GL_2(\mathbb{R})$, and thus all the local degrees are $+1$. Hence $\deg(\varphi) = k$, and so we are done (by homotopy invariance of the degree).

□

4.2. **Cell Complexes.**

In all of our computations so far for S^n, Σ_g, $\mathbb{C}P^k$, Klein bottle, etc, we have found that $H_*(X)$ has finite total rank, even though $C_*(X)$ is indecently large. For nice spaces, there is a smaller chain-level model which simplifies computations, called a *cellular complex*.

Definition 4.2. *A **cell complex** X is a space defined inductively as follows.*

(i) X_0 is a finite set,

(ii) Given X_{k-1}, define

$$X_k := X_{k-1} \cup \bigcup_{j \in I_k} D_j^k,$$

where D_j^k is a closed k-dimensional disc, I_k is an index set, attached via maps $\partial D_j^k \to X_{k-1}$ (so by this union we mean the images of D_j^k under such maps)

(iii) $X = \bigcup_{k \geq 0} X_k$, with the weak topology, i.e. $U \subset X$ is open if $U \cap X_k$ is open in X_k for all k.

*We call X_k the **k − skeleton** of X.*

Remark: We say that X is **finite-dimensional** if there are only finitely many different skeletons, i.e. if $X = X_N$ for some N.

We say that X is **finite** it is has only finitely many cells, i.e. if $X = X_N$ for some N and $|I_j| < \infty$ for all $0 \leq j \leq N$.

For us we will always have $|I_j| < \infty$ for each j, i.e. we only attach finitely many cells at each stage, unless we say otherwise.

Notation: If X is a cell complex with finitely many cells of dimension k, we call this number n_k.

Example 4.2. *Since $S^n = \{\text{point}\} \cup \overline{D}^n$, where D^n is attached via a constant map $\partial D^n = S^{n-1} \to \{\text{point}\}$. Thus S^n is cell complex with 1 0-cell and one n-cell.*

Example 4.3. *The second diagram in Figure 18 shows that $\Sigma_1 = T^2$ is a cell complex with 1 0-cell, 2 1-cells, and 1 2-cell. Similarly, the third diagram in Figure 18 enables us to work out the cell decomposition of Σ_2, or more generally Σ_g (which has 1 0-cell, $2g$ 1-cells, and 1 2-cell).*

Fact: If M is a smooth closed manifold (at least if $\dim(M) \neq 4$), then M admits the structure of a cell complex (this can be proved via Morse theory).

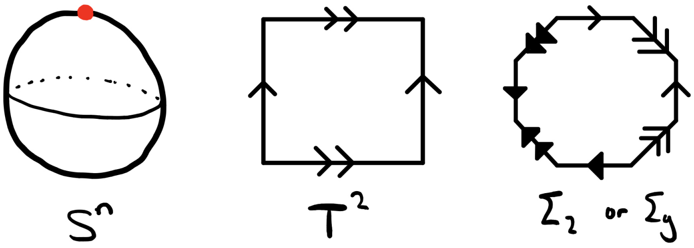


FIGURE 18. Illustrations of the cell complexes discussed in Example 4.2 and 4.3.

Example 4.4 (Wedge Products)**.** *Suppose X, Y are cell complexes, and $p \in X_0$, $q \in Y_0$. Then we can form $X \vee Y$ via*

$$X \vee Y := \frac{X \amalg Y}{p \sim q}$$

i.e. glue X, Y together at p and q. Then clearly $X \vee Y$ is another cell complex, and the complexes of $X \vee Y$ are simply those of X and those of Y (i.e. total number is the sum of those in X and those in Y), except for the 0-cells when we get one less, as we have glued to points (which are 0-cells) together.

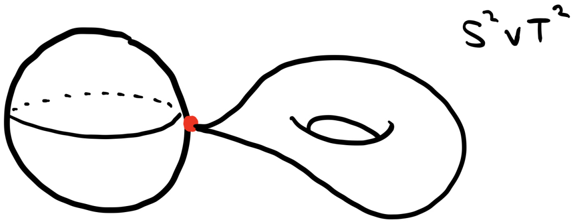


FIGURE 19. An illustration of a wedge product.

Example 4.5. *Suppose X, Y are cell complexes. Then $X \times Y$ has a* ***product cell structure****, with the* ***open cells*** *being the products of those in X, Y [here the* ***open cells*** *are the interiors of the D_j^k. Thus a cell complex is, by definition, the disjoint union of its open cells].*

We now work towards finding suitable abelian groups to define the cellular chain complex.

Lemma 4.1. *Suppose X is a cell complex and $A \subset X$ is a subcomplex. Then the pair (X, A) is a good pair.*

In particular, $H_*(X, A) \cong \tilde{H}_*(X/A)$, by Proposition 3.1.

Proof. See Example Sheet 2. □

Proposition 4.2. *Let $X = \cup_{k \geq 0} X_k$ be a* <u>connected</u> *cell complex. Then:*

(i) *$H_k(X_k, X_{k-1})$ is a free abelian group on the k—cells, and $H_i(X_k, X_{k-1}) = 0$ if $i \neq k$.*
(ii) *$H_*(X_k) = 0$ if $* > k$.*
(iii) *The inclusion map $X_k \hookrightarrow X$ induces an isomorphism $H_i(X_k) \xrightarrow{\cong} H_i(X)$ for $i < k$.*

Note: If X is a finite cell complex, then an easy induction shows that $H_*(X)$ is of finite total rank.

Proof. (i): We have $X_{k-1} \subset X_k$ is a subcomplex, and thus by Lemma 4.1 we have

$$H_*(X)j, X_{k-1}) \cong \tilde{H}_*(X_k/X_{k-1}).$$

But then we have (from how the k−skeletons were defined),

$$\frac{X_k}{X_{k-1}} \simeq \bigvee_{\alpha \in I_k} S^k$$

where I_k indexes the k−cells (this is simply because if we crush X_{k-1} to a point we crush all boundaries of the attached k−cells to a point. But then as they are all attached to X_{k-1}, the are all attached at a point after this crushing - think of crushing the boundary of a 2-disc in $\mathbb{R}^3$ to see this).

Now (i) follows from an easy application of Mayer-Vietoris.

(ii): Let us consider the l.e.s of the pair (X_k, X_{k-1}):

$$\cdots \longrightarrow H_{i+1}(X_k, X_{k-1}) \longrightarrow H_i(X_{k-1}) \longrightarrow H_i(X_k) \longrightarrow H_i(X_k, X_{k-1}) \longrightarrow \cdots$$

If $i > k$, then this comes

$$0 \longrightarrow H_i(X_{k-1}) \longrightarrow H_i(X_k) \longrightarrow 0$$

and thus by exactness $H_i(X_{k-1}) = H_i(X_i)$. Applying this equality by induction gives

$$H_i(X_{k-1}) \cong H_i(X_{k-1}) \cong \cdots \cong H_i(X_0) = 0$$

and so done.

(iii): Similarly to the above, if $i < k$ the l.e.s of the pair gives

$$H_i(X_k) \cong H_i(X_{k+1}) \cong \underset{\text{induction}}{\cdots} \cong H_i(X_N)$$

for $N > k$. So if X is finite dimensional, then we're done as $X_N = X$ for some N. But even if X is not finite dimensional, an element of $H_K(x)$ is a finite sum of simplicies, and so is represented by a chain with compact image in X. From Example Sheet 2, this means that it comes from an element in $H_k(X_N)$, for some N. But then our finite dimensional case says this is independent of N for large enough N, and so $H_i(X_N) \to H_i(X)$ is onto for $N > i$. Hence we are done.

□

Corollary 4.1. *We have*

(i) If X is a finite dimensional cell complex, then

$$H_i(X) = 0 \quad \textit{for} \quad i \notin \{0, 1, \ldots, \dim_{\mathbb{R}}(X)\}.$$

(ii) For any sequence $b_1, b_2, \ldots, \in \mathbb{N}$, $\exists$ a cell complex with $H_i(X) \cong \mathbb{Z}^{b_i}$ for all i.

Proof. (i): Clear by Proposition 4.2.

(ii): Take a suitable wedge of spheres of different dimensions to see this (i.e. b_i copies of S^i).

□

4.3. Cellular (co)homology.

We are now ready to define the cellular chain complex. To do this, we use the maps from the l.e.s of the pairs (X_k, X_{k-1}), which are shown in red in the diagram below.

Definition 4.3. *Let X be a cell complex with finitely many cells of each dimension. Then the **cellular chain complex** $C_*^{\text{cell}}(X)$ has **cellular chain groups** given by*

$$C_k^{\text{cell}}(X) := H_k(X_k, X_{k-1}) \quad (\cong \mathbb{Z}^{n_k} \text{ if } X \text{ has } n_k \text{ } k\text{—cells})$$

with the boundary map defined via the l.e.s of the pairs $(X_k, X_{k-1})_k$:

$$\begin{tikzcd}
& & {} \arrow[dr] & & \iddots & & & \\
& & & H_k(X_k) \arrow[ur] \arrow[dr, "i"] & & & & \iddots \\
\cdots \arrow[rr] & & H_{k+1}(X_{k+1},X_k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "d^{\text{cell}}"'] & & H_k(X_k,X_{k-1}) \arrow[rr, "d^{\text{cell}}"] \arrow[dr, "\partial_k"] & & H_{k-1}(X_{k-1},X_{k-2}) \arrow[ur] \arrow[r] & \cdots \\
& H_{k+1}(X_{k+1}) \arrow[ur] & & & & H_{k-1}(X_{k-1}) \arrow[ur, "i"] \arrow[dr] & & \\
& & & & {} \arrow[ur] & & \ddots &
\end{tikzcd}$$

i.e. the LHS red diagonals are from the l.e.s of (X_{k+1}, X_k), the blue diagonals are from the l.e.s of (X_k, X_{k-1}), etc, and both colours are used when they meet. So here ∂_k are the boundary maps from those l.e.s's whereas the i are the natural inclusion map from the l.e.s. Thus we define d^{cell} by what makes this commute, i.e.

$$d_k^{\text{cell}} := i \circ \partial_k.$$

Note: $d^{\text{cell}} \circ d^{\text{cell}} = 0$ as the composition includes two successive maps in the l.e.s of (X_k, X_{k-1}), i.e.

$$d_k^{\text{cell}} \circ d_{k+1}^{\text{cell}} = i \circ \underbrace{\partial_k \circ i}_{\text{both from blue line above}} \circ \partial_{k+1} = i \circ 0 \circ \partial_{k+1} = 0$$

since the l.e.s of the pair (X_k, X_{k-1}) is exact. Thus we can define the **cellular homology** via the homology of this chain complex, i.e.

$$H_*^{\text{cell}}(X) := H_*\left(C_*^{\text{cell}}(X), d^{\text{cell}}\right).$$

Proposition 4.3. *Cellular homology agrees with singular homology, i.e.*

$$H_*^{\text{cell}}(X) \cong H_*(X).$$

Proof. From Proposition 4.2(ii), we know that $H_*(X_k) = 0$ if $* > k$. Hence if we look at the l.e.s of the pair (X_k, X_{k-1}), we have at $* = k$,

$$\underbrace{H_{k+1}(X_k)}_{=0} \longrightarrow H_k(X_k) \longrightarrow H_k(X_k, X_{k-1}) \longrightarrow H_{k-1}(X_{k-1})$$

and then from Proposition 4.2(iii), we know $H_*(X_k) \cong H_*(X)$ if $* < k$, and so looking at the l.e.s of the pair (X_{k+1}, X_k) we have

$$H_{k+1}(X_{k+1}, X_k) \longrightarrow H_k(X_k) \longrightarrow \underbrace{H_k(X_{k+1})}_{\cong H_k(X)} \longrightarrow \underbrace{H_k(X_{k+1}, X_k)}_{=0}$$

where on the last erm we have used Proposition 4.2(i). Hence inserting this into the defining diagram for d_*^{cell}, we have the following diagram:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & H_k(X_{k-1}) = 0 \arrow[dr] & & H_k(X_{k+1}) = H_k(X) \arrow[ur] & & & \\
 & & H_k(X_k) \arrow[ur] \arrow[dr, "i"] & (\dagger) & & & \\
\cdots & H_{k+1}(X_{k+1}, X_k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "d^{\text{cell}}"] & & H_k(X_k, X_{k-1}) \arrow[rr, "d^{\text{cell}}"] \arrow[dr, "\partial_k"] & & H_{k-1}(X_{k-1}, X_{k-2}) & \cdots \\
 & & & & H_{k-1}(X_{k-1}) \arrow[ur, "i"] & (\star) & \\
 & & & H_{k-1}(X_{k-2}) = 0 \arrow[ur] & & &
\end{tikzcd}$$

Working along the diagonal arrows (which we know are exact) and using the first isomorphism theorem, we have (since several groups as shown as zero):

$$\begin{aligned}
H_*(X) \cong H_k(X_{k+1}) &= \frac{H_k(X_k)}{\operatorname{Im}(\partial_{k+1})} && \text{by exactness} \\
&= \frac{i(H_k(X_k))}{\operatorname{Im}(i \circ \partial_{k+1})} && \text{since } i \text{ is injective at } (\dagger) \text{ (by exactness)} \\
&= \frac{\ker(\partial_k)}{\operatorname{Im}(d_{k+1}^{\text{cell}})} && \text{by exactness of the pair } (X_k, X_{k-1}) \\
&= \frac{\ker(i \circ \partial_k)}{\operatorname{Im}(d_{k+1}^{\text{cell}})} && \text{since } i \text{ is injective at } (\star) \text{ (by exactness)} \\
&= \frac{\ker(\partial_k^{\text{cell}})}{\operatorname{Im}(d_{k+1}^{\text{cell}})} && \text{since } i \circ \partial_k = d_k^{\text{cell}} \\
&=: H_k^{\text{cell}}(X)
\end{aligned}$$

and so we are done.

□

Remark: $C^{\text{cell}}_*(X)$ is not naturally functorial under continuous maps. Instead, the continuous maps which are functorial here are called **cellular maps**. C^{cell}_* is functorial under such maps.

Definition 4.4. *A map $f : X \to Y$ of cell complexes is called a* ***cellular map*** *if it preserves the cellular structure, i.e. if $f(X_k) \subset Y_k$ for all k.*

[The thing with cell complexes is that we have natural spaces to consider, namely the relative homology of one complex with respect to another. This is what we use to form the cellular chain complex and thus define cellular homology.]

Corollary 4.2. *Let X be a cell complex. Then:*

(i) Uf X has a cell structure with finitely many k-cells, then $H_k(X)$ is a finitely generated abelian group of rank $\leq n_k$ (recall that n_k was the number of k—cells in X).

(ii) If $H_k(X) \neq 0$, any cell structure on X must have $n_k \geq rank(H_k(X))$, i.e. at least this many k—cells.

(iii) If X is compact, $H_*(X)$ is a finitely generated abelian group, and $H_*(X, \mathbb{Q})$ is a finite-dimensional vector space.

(iv) If X has a cell structure $X = \bigcup_{k \geq 0} X_k$ with only even-dimensional cells, then

$$H_*(X) = C^{cell}_*(X) \qquad \text{[for the \underline{given} cell structure].}$$

Proof. These are left as exercises. □

Note: $\mathbb{C}P^n$ is a space as in (iv) above. Similarly the complex Grassmannians, $\mathrm{Gr}(k, \mathbb{C}^n)$, are as well.

So how do we compute d^{cell}? It turns out we can use what we know about degrees of maps $S^n \to S^n$ to do this.

Lemma 4.2. *Let $k \geq 2$. Suppose X is a cell complex with n_k k—cells for each k. For $d^{\text{cell}} : C^{\text{cell}}_k(X) \to C^{\text{cell}}_{k-1}(X)$, let $[D^k_\alpha]$ be a k—cell. Then we know (as we have a basis) we can write*

$$d^{\text{cell}}_k([D^k_\alpha]) = \sum_\beta d_{\alpha\beta} \cdot [D^{k-1}_\beta]$$

where the sum is over $(k-1)$—cells. Consider the following composite of maps

$$\begin{tikzcd} S^{k-1}_\alpha \arrow[r, "\text{attaching}", "\text{map of } D^k_\alpha"'] & X_{k-1} \arrow[r] & \frac{X_{k-1}}{X_{k-2}} \arrow[r] & \bigvee_{\gamma:(k-1)-cell} S^{k-1} \arrow[r] & S^{k-1}_\beta \end{tikzcd}$$

where the final map is just collapsing all other $(k-1)$—cells except the β'th. Thus this gives a map $S^{k-1}_\alpha \to S^{k-1}_\beta$, and so has a degree. This degree is $d_{\alpha\beta}$.

Remark: For $d_{\alpha\beta}$ to be well-defined (and not just up to a sign) we need to fix isomorphisms $H_{k-1}(S^{k-1}_\alpha) \cong \mathbb{Z}$, and similarly for β. For instance, if $\partial D^k_\alpha \cong S^{k-1} \subset \mathbb{R}^k$ is a given isomorphism, we can use this to get a canonical generator.

Proof. We can write down the following diagram, with black spaces/arrows being ones we know and red ones being ones we include to help us.

$$\begin{tikzcd}
\overset{\cong\mathbb{Z}}{H_k(D^k_\alpha, \partial D^k_\alpha)} \arrow[r, "\cong"] \arrow[d, "\varphi_\alpha"] & H_{k-1}(\partial D^k_\alpha) \arrow[r, "\times d_{\alpha\beta}"] \arrow[d, "\varphi_\alpha|_{\partial D^k_\alpha}"] & H_k(S^{k-1}_\beta) \\
H_k(X_k, X_{k-1}) \arrow[r] \arrow[dr, "d_k^{\text{cell}}"'] & H_{k-1}(X_{k-1}) \arrow[r] \arrow[d] & H_{k-1}(X_{k-1}/X_{k-2}) \arrow[u, "\text{collapse}"'] \arrow[d, "\cong"] \\
 & H_{k-1}(X_{k-1}, X_{k-2}) \arrow[r] & H_k\left(\frac{X_{k-1}}{X_{k-2}}, \{\text{point}\}\right)
\end{tikzcd}$$

where the bottom right red isomorphism is because for $k \geq 2$ reduced homology is just the homology. Here φ_α is the attaching map of the $k-$cell. Then one can see that this diagram commutes, and then this proves the lemma.

□

Example 4.6 (Real Projective Space)**.** *We have* $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup \mathbb{R}^n = \cdots = \mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \cdots \cup \{point\}$*, and thus this has a cell structure with 1 cell of each degree for degrees* $0 \leq k \leq n$*.*

Note that $S^n/\{\pm 1\} = \text{hemisphere}/\{\pm 1 \text{ on } \partial(\text{hemisphere})\}$*. So the cell complex is:*

$$C^{\text{cell}}_* : \underset{\text{deg}}{0} \to \underset{n}{\mathbb{Z}} \to \underset{n-1}{\mathbb{Z}} \to \cdots \to \underset{1}{\mathbb{Z}} \to \underset{0}{\mathbb{Z}} \to 0$$

and

$$d_k^{\text{cell}} : \partial D^k \to \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}$$

where the red map is generically 2:1 (as it comes from the canonical double cover), and the local maps at the two pre-images are homeomorphisms differing by the antipodal map of ∂D^k_α*.*

Using the expression of the degree of a map of spheres as a sum of its local degrees (Proposition 4.1) one sees:

$$d_k^{\text{cell}} : \mathbb{Z} \to \mathbb{Z}$$

is multiplication by $1 + (-1)^k$ *(up to a sign). So hence we have two different situations depending on the parity of n:*

$$\begin{aligned}
&n \text{ even:} \quad 0 \to \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \\
&n \text{ odd:} \quad 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\pm 2} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0
\end{aligned}$$

where the numbers are the degrees of the maps, and thus we see

$$H_*(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ and } n \text{ odd} \\ \mathbb{Z}_2 & \text{if } 0 < * < n \text{ and } * \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

whereas we see

$$H_*(\mathbb{R}P^n;\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{if } 0\leq *\leq n\\ 0 & \text{otherwise.}\end{cases}$$

Hence working with homology over a different abelian group can make things simpler.

Remark: There is also **cellular cohomology**. Here, $C^k_{\text{cell}}(X):=H^k(X_k,X_{k-1})$. Then:

(i) There is a differential $d^k_{\text{cell}}:C^k_{\text{cell}}(X)\to C^{k+1}_{\text{cell}}(X)$ obtained in an analogous way for the l.e.s of pairs, and the identification between $H^*_{\text{cell}}(X)$ and $H^*(X)$ follows in the same way [**Exercise** to check/think about].

(ii) Again, H^*_{cell} is natural only under cellular maps at the cochain level.

(iii) One can check that (as in the above example)

$$H_*(\mathbb{R}P^2;\mathbb{Z})=H_*(\mathbb{Z}\xrightarrow{\pm 2}\mathbb{Z}\to 0)=\begin{cases}\mathbb{Z} & \text{if } *=0\\ \mathbb{Z}_2 & \text{if } *=1\\ 0 & \text{otherwise}\end{cases}$$

whilst

$$H^*(\mathbb{R}P^2;\mathbb{Z})=H^*(\mathbb{Z}\overset{\pm 2}{\leftarrow}\mathbb{Z}\overset{0}{\leftarrow}\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{if } *=0\\ \mathbb{Z}_2 & \text{if } *=2\\ 0 & \text{otherwise}\end{cases}$$

and thus we see we have an example where $H^*\neq H_*$.

If we look at the above remark (iii), we see that the cohomology doesn't always agree with the homology. However we do see that there is a relation - the torsion group $\mathbb{Z}_2$ from the homology is bumped up a degree in the cohomology. It turns out that this is true in general for finite cell complexes: to find the cohomology (w.r.t $\mathbb{Z}$) $H^i(X,\mathbb{Z})$, we take the freely generated part of $H_i(X,\mathbb{Z})$, and the torsion part of $H_{i-1}(X,\mathbb{Z})$ and combine them. Put another way, look at the homology, fix all parts of the form $\mathbb{Z}^n$, and shift all other torsion parts up one place. This is what the following proposition tells us.

Proposition 4.4 (Relation between homology and cohomology (over $\mathbb{Z}$)). *Let X be a finite cell complex. Then:*

$$H^i(X,\mathbb{Z})=\frac{H_i(X,\mathbb{Z})}{\text{Torsion}}\oplus\text{Torsion}(H_{i-1}(X,\mathbb{Z})).$$

Note: Recall that for a finitely generated abelian group G we have

$$\text{Torsion}(G)=\{\text{subgroup of elements of finite order}\}$$

e.g. if $G\cong\mathbb{Z}^c\oplus\mathbb{Z}_{r_1}\oplus\cdots\oplus\mathbb{Z}_{r_k}$, then $\text{Tor}(G)=\mathbb{Z}_{r_1}\oplus\cdots\oplus\mathbb{Z}_{r_k}$.

Proof. We break the proof of Proposition 4.4 up into two steps.

Step 1: We first claim that the cellular chain complex is the dual of the cellular chain complex.

A question on example sheet 2 gives that for all spaces X and pairs (X,A), $\exists$ a natural surjective homomorphism $H^n(X,A)\xrightarrow{\eta}\text{Hom}(H_n(X,A),\mathbb{Z})$. Then we consider the cellular cochain complex and the dual of the cellular chain complex, with the η maps providing a means to travel between the two:

$$\begin{tikzcd}
H^k_{\text{cell}}(X_k,X_{k-1}) \arrow[r] \arrow[rr, bend left, "d^k_{\text{cell}}"] \arrow[d, "\eta", "\cong"'] & H^k(X_k) \arrow[r] \arrow[d, "\eta"] & H^{k+1}(X_{k+1},X_k) \arrow[d, "\eta", "\cong"'] \\
\text{Hom}(H_k(X_k,X_{k-1});\mathbb{Z}) \arrow[r] \arrow[rr, bend right, "(d_k^{\text{cell}})^*"'] & \text{Hom}(H_k(X_k),\mathbb{Z}) \arrow[r] & \text{Hom}(H_{k+1}(X_{k+1},X_k),\mathbb{Z})
\end{tikzcd}$$

i.e. the top line is the cellular cochain complex, whilst the bottom is the dual of the cellular chain complex. The η maps enable us to travel between the two. We know that the outer η maps are isomorphisms since they are always surjective and since $H(X,A)$ is a free group in these cases [from Proposition 4.2(i)].

We want to show that this diagram commutes, as then we will be done. The left hand square commutes by naturality of η. The right hand square commutes by naturality of the l.e.s of pairs and the construction of η. Hence the whole diagram commutes, and so we see

$$H^k(X_k)\cong\text{Hom}(H_k(X_k),\mathbb{Z})$$

which finishes step 1.

Step 2: We claim that whenever C_* is a chain complex with finitely generated chain groups, and $C^*=\text{Hom}(C_*,\mathbb{Z})$ (as we now know this is true) is the dual cochain complex, then the conclusion of the proposition holds.

We split C_* into collections of s.e.s's:

$$0\longrightarrow B_n\longrightarrow Z_n\longrightarrow H_n(C_*)\longrightarrow 0$$

$$0\longrightarrow Z_n\longrightarrow C_n\longrightarrow B_{n-1}\longrightarrow 0$$

where $B_n=\text{Im}(d_n)$ and $Z_n=\ker(d_n)$. In the latter s.e.s, all groups are free and so we can (non-canonically) split it, i.e. $\exists$ maps $\iota_n : B_{n-1}\to C_n$ such that $d_n\circ\iota_n=\text{id}|_{B_{n-1}}$. Such a splitting induces isomorphisms $C_n\cong\mathbb{Z}_n\oplus B_{n-1}$. Now C_* can be written as

$$Z_{n+1}\oplus B_n\xrightarrow{d_{n+1}}Z_n\oplus B_{n-1}\xrightarrow{d_n}Z_{n-1}\oplus B_{n-2}\to\cdots$$

and then we can see that the map $d_{n+1} : Z_{n+1} \oplus B_n \to Z_n \oplus B_{n-1}$ reduces to a map $d_{n+1} : B_n \to Z_n$ (as it is zero on $Z_{n+1} = \ker(d_{n+1})$ and maps $B_n = \mathrm{Im}(d_n)$ to Z_n).

Hence C_* breaks into a direct sum of two-term complexes of the form

$$0 \longrightarrow B_n \xrightarrow{\alpha_n} Z_n \longrightarrow 0.$$

Smith-normal form then says that ∃ a $\mathbb{Z}$—linear change of basis such that α_n has matrix (as chain groups are finitely generated):

$$\begin{pmatrix} d_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & d_i & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

where $d_1|d_2$, $d_2|d_3$, etc. So now C_* break into a sum of complexes of the form

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \qquad \text{or} \qquad 0 \to \mathbb{Z} \xrightarrow{d} \mathbb{Z} \to 0$$

for $d \in \mathbb{Z}$. From these, we see that $H_*(C_*)$ and $H^*(\mathrm{Hom}(C_*), \mathbb{Z})$ are related in the required way [**Exercise** to check - dualising the latter equation to $0 \leftarrow \mathbb{Z} \xleftarrow{d} \mathbb{Z} \leftarrow 0$ and using properties of d shows that the torsion is shifted one degree up.]

□

Remark: The **universal coefficients theorem** says ∃ a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \xrightarrow{\eta} \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

where

$$\mathrm{Ext}(H, G) := \frac{\{\text{s.e.s's } 0 \to G \to J \to H \to 0\}}{\text{natural notion of isomorphism of s.e.s's}}.$$

Remark: The above Proposition 4.4 is true if we just assume that the $H_*(X)$ are finitely generated (so this wouldn't work for, e.g. $C_*(X)$), but this takes more work to show.

We now talk briefly about the Euler characteristic, since no discussion about cell complexes would be complete without it.

4.4. **Euler Characteristic.**

The Euler characteristic is the simplest homotopy invariant quantity which is computable from cell complexes.

Let X be a finite cell complex.

Definition 4.5. *The* ***Euler characteristic*** *of X is:*

$$\chi(X) := \sum_{k\geq 0}(-1)^k n_k$$

where n_k = number of k—cells in X.

Lemma 4.3. *We have*

$$\chi(X) = \sum_{k\geq 0}(-1)^k \text{rank}_{\mathbb{Z}}(H_k(X,\mathbb{Z})) = \sum_{k\geq 0}(-1)^k \dim_{\mathbb{F}}(H_k(X,\mathbb{F}))$$

for any field $\mathbb{F}$.

In particular, as these ranks are homotopy invariant, so is $\chi(X)$, and thus the Euler characteristic only depends on the homotopy class of X and not the particular cell structure of X we use to compute it.

Proof. Recall from the proof of Proposition 4.4 we had

$$0 \to B_n \to Z_n \to H_n(C_*) \to 0 \quad \text{and} \quad 0 \to Z_n \to C_n \to B_{n-1} \to 0$$

which were s.e.s's into which $C_*^{\text{cell}}(X)$ split. Then we have from exactness in the latter s.e.s:

$$n_k = \text{rank}(C_k) = \text{rank}(Z_k) + \text{rank}(B_{k-1})$$

and exactness in the first s.e.s gives

$$\text{rank}_{\mathbb{Z}}(H_k(C_*)) = \text{rank}(Z_k) - \text{rank}(B_k)$$

and so if we write $z_k = \text{rank}(Z_k)$, $b_k = \text{rank}(B_k)$, then these say

$$n_k = z_k + b_{k-1} \quad \text{and} \quad \text{rank}_{\mathbb{Z}}(H_k(C_*)) = z_k - b_k$$

and so writing $b_{-1} = 0$, we get

$$\begin{aligned}\chi(X) = \sum_{k\geq 0}(-1)^k n_k &= \sum_{k\geq 0}(-1)^k(z_k + b_{k-1})\\ &= \sum_{k\geq 0}(-1)^k(z_k - b_k)\\ &= \sum_{k\geq 0}(-1)^k\text{rank}_{\mathbb{Z}}(H_k(C_*))\end{aligned}$$

as desired.

For working over a general field, just do as above, noting that $\text{rank}(C_k) = \dim_{\mathbb{F}}(C_k \oplus \mathbb{F})$.

□

Example 4.7. *We can calculate that $\chi(S^4) = 2$ and $\chi(\mathbb{C}P^2) = 3$, and thus by the homotopy invariance of $\chi(X)$ above, we have $S^4 \not\simeq \mathbb{C}P^2$.*

Note: Using the original definition of $\chi(X)$ in terms of an alternating sum of numbers of cells of each degree, just by counting cells we get that

$$\chi(A\times B)=\chi(A)\chi(B) \qquad \text{and} \qquad \chi(A\cup B)=\chi(A)+\chi(B)-\chi(A\cap B)$$

if A,B are subcomplexes of X (e.g. $X=A\cup B$). Thus simple counting on the level of cell complexes gives something non-trivial on the level of homology.

5. AXIOMATIES

Definition 5.1. *An assignment $h_* : (X,A) \mapsto h_*(X,A)$ of pairs (X,A) (with X a space and $A \subset X$ a subspace) to graded abelian groups $h_*(X,A)$ is called a* ***generalised homology theory*** *(GHT) if it satisfies the following properties:*

- *(i)* ***Functorality****: if $f : (X,A) \to (Y,B)$ is a map of pairs, then it induces a map $f_* : h_*(X,A) \to h_*(Y,B)$ in such a way that*
$$(\mathrm{id})_* = \mathrm{id} \quad \textit{and} \quad (f \circ g)_* = f_* \circ g_*$$
- *(ii)* ***Homotopy invariance:*** *if $f : (X,A) \to (Y,B)$ and $g : (X,A) \to (Y,B)$ are homotopic as maps of pairs, then $f_* = g_*$ as maps $h_*(X,A) \to H_*(Y,B)$.*
- *(iii)* ***Exact sequences:*** *writing $h_*(X) := h_*(X,\emptyset)$, where $\emptyset$ is the empty subspace, then there is a l.e.s*
$$\cdots \to h_i(A) \to h_i(X) \to h_i(X,A) \to h_{i-1}(A) \to h_{i-1}(X) \to \cdots$$
which are themselves functorial under maps.
- *(iv)* ***Excision:*** *if $\overline{Z} \subset Int(A)$, the inclusion map induces an isomorphism*
$$h_*(X\backslash Z, A\backslash Z) \xrightarrow{\cong} h_*(X,A).$$
- *(v)* ***Unions:*** *if $X = \amalg_\alpha X_\alpha$ is a union of path-components, then*
$$h_*\left(\amalg_\alpha X_\alpha\right) = \bigoplus_\alpha h_*(X_\alpha).$$

Intuitively, a generalised homology theory is one which obeys these 5 properties which we know hold for singular homology.

Proposition 5.1. *Let h_* and k_* be generalised homology theories, and let $\Phi : h_* \to k_*$ be a natural transformation*[iii]*. Then if $\Phi_{\{point\}} : h_*(\{point\}) \to k_*(\{point\})$ is an isomorphism, then $\Phi_{(X,A)}$ is an isomorphism for all pairs (cell complex, subcomplex, etc).*

Note: This does not mean that $h_*(\{\text{point}\})$ formally determines $h_*(X,A)$ from the axioms, but rahter, two theories agreeing on a point have the same indeterminacy.

Proof. Let X be a cell complex. Then if $\dim(X) = 0$, X is a finite or discrete set, and the unions axiom (v) says that Φ_α is an isomorphism.

Inductively, assume that $\Phi_{(X,A)}$ is an isomorphism for all cell complex pairs, where $\dim(X) \leq n-1$. Then take $X = X_n$ be n-dimensional. Consider the diagram (from the exact sequences axiom, using

---

[iii] i.e. we have $\Phi_{(X,A)} : h_*(X,A) \to k_*(X,A)$ for all pairs (X,A) and this map is compatible with all the structure.

the Φ maps to pass between them):

$$\begin{tikzcd}
h_{*+1}(X_n,X_{n-1}) \arrow[r] \arrow[d, "\Phi"] & h_*(X_{n-1}) \arrow[r] \arrow[d, "\Phi", "\cong"'] & h_*(X_n) \arrow[r] \arrow[d, "\Phi"] & h_*(X_n,X_{n-1}) \arrow[r] \arrow[d, "\Phi"] & h_{*-1}(X_{n-1}) \arrow[r] \arrow[d, "\Phi", "\cong"'] & \cdots \\
k_{*+1}(X_n,X_{n-1}) \arrow[r] & k_*(X_{n-1}) \arrow[r] & k_*(X_n) \arrow[r] & k_*(X_n,X_{n-1}) \arrow[r] & k_{*-1}(X_{n-1}) \arrow[r] & \cdots
\end{tikzcd}$$

where the red isomorphisms we know by induction. Thus if we can show that if the blue Φ maps, i.e. $\Phi_{(X_n,X_{n-1})}$, are isomorphisms, then the middle map would be an isomorphism by the 5-Lemma, and so we would have proven the rest for all finite cell complexes by induction.

Thus it suffices to show that $\Phi_{(X_n,X_{n-1})} : h_*(X_n,X_{n-1}) \to k_*(X_n,X_{n-1})$ is an isomorphism. But by the excision axiom (iv) and the unions axiom (v) we have

$$h_*(X_n,X_{n-1}) \overset{(iv)}{\cong} h_*\left(\amalg_\alpha D^n_\alpha, \amalg_\alpha \partial D^n_\alpha\right) \overset{(v)}{\cong} \bigoplus_\alpha h_*\left(D^n_\alpha, \partial D^n_\alpha\right)$$

and similarly this holds for k_*. So hence if we just work with each term in this direct sum, to show that $\Phi_{(X_n,X_{n-1})}$ is an isomorphism it suffices to prove that $\Phi_{(D^n,\partial D^n)}$ is an isomorphism.

But now:

$$\begin{tikzcd}
h_*(\partial D^n) \arrow[r] \arrow[d, "\Phi", "\cong"'] & h_*(D^n) \arrow[r] \arrow[d, "\Phi", "\cong"'] & h_*(D^n,\partial D^n) \arrow[r] \arrow[d] & h_{*-1}(\partial D^n) \arrow[r] \arrow[d, "\Phi", "\cong"'] & h_{*-1}(D^n) \arrow[r] \arrow[d, "\Phi", "\cong"'] & \cdots \\
k_*(\partial D^n) \arrow[r] & k_*(D^n) \arrow[r] & k_*(D^n,\partial D^n) \arrow[r] & k_{*-1}(\partial D^n) \arrow[r] & k_{*-1}(D^n) \arrow[r] & \cdots
\end{tikzcd}$$

where the red isomorphisms are by induction, as ∂D^n is an $(n-1)$-dimensional cell-complex, and the blue isomorphisms are by the homotopy invariance axiom (ii). Thus by the 5-Lemma we see that $\Phi_{(D^n,\partial D^n)}$ is an isomorphism, and so this proves the result for finite cell complexes by induction.

For a general cell complex we just need to use that: "$H_i(X_n) \to H_i(X)$ is an isomorphism once $n > i$" to reduce to the finite dimensional case. Then we are done.

□

Remark: A **generalised cohomology theory** is an assignment $(X,A) \mapsto h^*(X,A)$ such that:

(i) it is contravariant functorially: i.e. $f : (X,A) \to (Y,B)$ induces a map $f^* : h^*(Y,B) \to h^*(X,A)$ with the standard properties

(ii) Homotopy invariance

(iii) Existence of a l.e.s

(iv) Excision holds

(v) For unions we have

$$h^*\left(\amalg_\alpha X_\alpha\right) = \prod_\alpha h^*(X_\alpha)$$

(notice that we have a direct product here instead of a direct sum which is what we had for homology).

Note that then an analogous proposition for generalised cohomology theories to the above holds.

Example 5.1. *Singular (co)homology is an example of a generalised (co)homology theory, and we have*

$$h_*(\{point\}) := \begin{cases} \mathbb{Z} & if * = 0 \\ 0 & otherwise. \end{cases}$$

Example 5.2 (K-Theory)**.** *Consider the group:*

$$K_0(X) := \left(\frac{\{Vector\ bundles\ on\ X\}}{Isomorphism}, \oplus \right)$$

so that $\oplus$ is the group law. Then define:

$$K_i(X) := K_0(\Sigma^i X)$$

where Σ is the ***suspension operator****, and $\Sigma^i = \underbrace{\Sigma \circ \cdots \circ \Sigma}_{i\ times}$. Then this is a generalised homology theory, called* ***K-Theory****, and we have*

$$K_*(\{point\}) = \begin{cases} \mathbb{Z} & if\ *\ is\ even \\ 0 & otherwise. \end{cases}$$

Example 5.3 (Stable Homotopy Theory)**.** *There are natural maps*

$$\cdots \to \pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \cdots$$

which eventually become isomorphisms. Then we can define

$$\pi_i^{st}(X) := \lim_{k \to \infty} \pi_{i+k}(\Sigma^k X).$$

However, π_*^{st} of spheres is unknown!

Remark: If $h_*(X, A)$ is a homology of a chain complex, then it is singular homology modulo something silly (like multiplication by -1), i.e. it is

$$(\text{singular homology}) \otimes h_*(\{\text{point}\}).$$

6. THE CUP PRODUCT

The key feature of cohomology as opposed to homology is that cohomology naturally has a ring structure, whilst homology just has a group structure. This is essentially because we can multiply linear maps together. The multiplicative structure is called the **cup product**. We first define it on elements of the cochain group before seeing that it in fact defines a map on cohomology.

Definition 6.1. *If $\varphi \in C^k(X)$ and $\psi \in C^l(X)$, we define their* **cup product**, *$\varphi \cdot \psi \in C^{k+l}(X)$ (sometimes denoted $\varphi \smile \psi$) by:*

$$(\varphi \cdot \psi)[v_0, \ldots, v_{n+k}] := \varphi([v_0, \ldots, v_k]) \cdot \psi([v_k, \ldots, v_{k+l}]).$$

Arguably the cup product has a very natural definition. The cup product will give the multiplicative structure on cohomology to make it into a ring. More generally, $C^*(X;R)$ can be made into a ring as above if R is a ring - multiplication on the RHS is jsut multiplication in the ring. We will always take R to be a commutative ring in this course.

Lemma 6.1 (Leibniz Rule for Cup Product). *For any $\varphi \in C^k(X)$, $\psi : C^l(X)$ and $d^* : C^* \to C^{*+1}$ we have:*

$$d^*(\varphi \cdot \psi) = (d^*\varphi) \cdot \psi + (-1)^k \varphi \cdot (d^*\psi).$$

Proof. We have:

$$(\dagger) \qquad ((d^*\varphi) \cdot \psi)[v_0, \ldots, v_{k+l+1}] = \sum_i (-1)^i \varphi[v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}] \psi[v_{k+1}, \ldots, v_{k+l+1}]$$

and

$$(\dagger\dagger) \qquad (-1)^k (\varphi \cdot d^*(\psi))[v_0, \ldots, v_{k+l+1}] = \varphi[v_0, \ldots, v_k] \sum_{i=k}^{k+l+1} (-1)^i \psi[v_k, \ldots, \hat{v}_i, \ldots, v_{k+l+1}].$$

The last term in (†) and the first term in (††) cancel: they have signs $(-1)^{k+1}$ and $(-1)^k$ respectively. All the remaining terms yield:

$$d^*(\varphi \cdot \psi)[v_0, \ldots, v_{k+l+1}] = (\varphi \cdot \psi) \left(\sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_{k+l+1}] \right).$$

□

Corollary 6.1. *The cup product descends to a map on $H^*(X)$, i.e. it induces a well-defined product operation*

$$H^k(X) \times H^l(X) \to H^{k+l}(X)$$

which makes $H^*(X)$ into a graded, unital ring.

Proof. If $\varphi \in C^k(X)$ and $\psi \in C^l(X)$ are closed, then certainly by the Leibniz rule for the cup product we have

$$d^*(\varphi \cdot \psi) = 0$$

and so $\varphi \cdot \psi$ defines an element of $H^{k+l}(X)$.

One can then check that this only depends on the cohomology class $[\varphi] \in H^k(X)$ and $[\psi] \in H^l(X)$. So hence this is a well-defined product on cohomology, making it into a ring.

Moreover, if $1 \in C^0(X)$ is the cochain such that $1(p) := 1$ for all $p \in X$ (recall that $C_0(X)$ is the free abelian group on points) then we see $d^*(1) = 0$, and so 1 is the unit for the cup product (at the chain level - and so descends to the unit in cohomology). So hence $(H^*(X), \cdot)$ is unital.

☐

Remark: The cup product is associative: if $\varphi \in C^k$, $\psi \in C^l$ and $\delta \in C^r$, then:

$$\varphi \cdot (\psi \cdot \delta) = (\varphi \cdot \psi) \cdot \delta \in C^{k+l+r}.$$

Remark: If $f : X \to Y$, then the induced map $f^* : C^*(Y) \to C^*(X)$ satisfies

$$f^*(\varphi \cdot \psi) = f^*(\varphi) \cdot f^*(\psi)$$

i.e. $f^* : H^*(Y) \to H^*(X)$ is a unital ring homomorphism.

Both of the above remarks can be checked immediately from the definition [**Exercise** to check].

Given X, Y, we can define a **cross product** $\times : H^*(X) \otimes H^l(Y) \to H^{k+l}(X \times Y)$ by:

$$(\varphi, \psi) \longmapsto (\mathrm{pr}_X^* \varphi) \cdot (\mathrm{pr}_Y^* \psi)$$

where $\mathrm{pr}_X : X \times Y \to X$, $\mathrm{pr}_Y : X \times Y \to Y$ are the projections.

In particular, if $\Delta : X \to X \times X$ is the diagonal map $x \mapsto (x, x)$, then we have a composition

$$H^k(X) \otimes H^l(X) \xrightarrow{\times} H^{k+l}(X \times X) \xrightarrow{\Delta^*} H^{k+l}(X)$$

which <u>is</u> the cup product [**Exercise** to check]. Hence the cup product is a special case of a cross product.

Essentially, contravariance of cohomology (instead of covariance for homology) and the existence of a god-given map $\Delta : X \to X \times X$ (as there is no god-given map $X \times X \to X$ - which projection do we take?) is what enables us to define the cup product and make cohomology into a ring. Cohomology therefore is in some sense a deeper object than homology.

Theorem 6.1 (Künneth Theorem). *Let X, Y both be a compact cell complexes, and suppose also that $H^i(Y)$ is free for each i. Then the cross-product induces an isomorphism $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$, i.e.*

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\cong} H^n(X \times Y).$$

Proof. Regard Y as fixed, and consider the two functors of pairs (X,A) given by:

$$(X,A) \longmapsto h^*(X,A) := H^*(X,A) \otimes H^*(Y)$$

$$(X,A) \longmapsto k^*(X,A) := H^*(X \times Y, A \times Y).$$

Now if $\varphi \in C^k(X) \cap C^k(X,A)$, i.e. φ vanishes on $C_k(A)$, and if $\psi \in C^l(Y)$, then $\varphi \times \psi \in C^{k+l}(X \times Y)$ and $\varphi \times \psi$ vanishes on $A \times Y$ (since we feed the front k—face into φ, and that lies in A (or at least, projects to A). So hence we see $\varphi \times \psi \in C^{k+l}(X \times Y, A \times Y)$. [So crossing with a relative cochain gives another relative cochain.]

Therefore we see that the cross-product defines a map

$$H^*(X,A) \otimes H^*(Y) \to H^*(X \times Y, A \times Y).$$

Write $\Phi_{(X,A)}$ for this cross-product induced map $h^*(X,A) \to k^*(X,A)$.

Now if $(X,A) = (\{\text{point}\}, \emptyset)$, both h^* and k^* give $H^*(Y)$, and the map $\Phi_{(\{\text{point}\},\emptyset)}$ is an isomorphism.

So <u>if</u> h^* and k^* are both generalised cohomology theories and $\Phi_{(\{\text{point}\},\emptyset)}$ is a natural isomorphism, then by the cohomology variant of Proposition 5.1 we see that $\Phi_{(X,A)}$ is an isomorphism for all (X,A).

It is easy to check [**Exercise**] that all the generalised cohomology theory axioms are satisfied for k^*. For h^*, one needs to be careful with the l.e.s axiom. The key point here is that $\otimes$(free module) (i.e. tensoring with a free module on a l.e.s) <u>does</u> preserve exactness, and similarly

$$\left(\prod_\alpha M_\alpha\right) \otimes N = \prod_\alpha (M_\alpha \otimes N)$$

agree when N is finitely generated and free. To hence h^* is a generalised cohomology theory.

Finally, to see that Φ is natural, we know the cup (and hence cross) product behave well with respect to maps of spaces. One therefore reduces to checking that the diagram

$$\begin{tikzcd}
H^k(A) \otimes H^l(Y) \arrow[r] \arrow[d, "\Phi"] & H^{k+1}(X,A) \otimes H^l(Y) \arrow[d, "\Phi"] \\
H^{k+l}(A \times Y) \arrow[r] & H^{k+l+1}(X \times Y, A \times Y)
\end{tikzcd}$$

commutes [**Exercise** to check]. Then we are done.

□

Remark: If we work over a field $\mathbb{F}$, $H^*(Y,\mathbb{F})$ is automatically free and so the Künneth formula always applies.

The other key feature of the cup product (aside from the Künneth theorem) is that it is graded commutative.

Proposition 6.1 (Graded Commutativity of Cup Product). *Suppose $\varphi \in H^k(X)$ and $\psi \in H^l(X)$. Then:*

$$\varphi \cdot \psi = (-1)^{kl} \psi \cdot \varphi \qquad \in H^{k+l}(X).$$

Note: The cup product is associated at the chain level, but the above identity only holds on cohomology.

Proof. Momentarily. ☐

Example 6.1. *Consider the cohomology of S^1, which we know to be*

$$H^*(S^1) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let x be a generator of $H^1(S^1)$. Then $x^2 = x \cdot x = 0$ since $x \cdot x \in H^2(S^1) = \{0\}$. [Recall $1 \in H^0(S^1)$ is a unit.]

By the Künneth formula,

$$H^*(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$$

which is naturally $H^*(S^1) \otimes H^*(S^1)$. Label the degree 1 generators of the two $H^1(S^1)$ factors here x_1, x_2. Then we know that $x_1^2 = x_2^2 = 0$ by naturality, and $x_1 \cdot x_2 = -x_2 \cdot x_1$ by graded commutativity (Proposition 6.1), and this is non-zero and generates $H^2(T^2) = H^2(S^1 \times S^1) = H^1(S^1) \otimes H^1(S^1)$.

The upshot is that we have a nice generating set:

$$H^*(T^2) \cong \left\langle x_1, x_2 \mid x_i^2 = 0,\ x_1 x_2 = -x_2 x_1 \right\rangle = \bigwedge^{*} \langle x_1, x_2 \rangle$$

and this is an exterior algebra.

The generalisation turns out to be:

$$H^*(T^n) \cong \bigwedge \underbrace{\langle x_1, \ldots, x_n \rangle}_{\text{all in degree 1}} \cong \bigwedge^{*} H^1(T^n).$$

Corollary 6.2. *Let $n \geq 2$. Then if $f : S^n \to T^n$ is any map, then f has zero degree, i.e. f induces the zero map $\underset{\cong\mathbb{Z}}{H^n(T^n)} \to \underset{\cong\mathbb{Z}}{H^n(S^n)}$.*

Proof. This is hard to show before we have a ring structure on cohomology. But now that we do, we know from the above example that the generator of $H^n(T^n)$ is $x_1 \cdot x_2 \cdots x_n$, with $x_i \in H^1(T^n)$. But then we know $f^* : H^1(T^n) \to H^1(S^n)$ is a ring homomorphism, and as $n \geq 2$ we have $H^1(S^n) = \{0\}$ and so $f^*(x_i) = 0$ for each i. Hence

$$f^*(x_1 \cdots x_n) = f^*(x_1) \cdots f^*(x_n) = 0$$

and thus $f^* : H^n(T^n) \to H^n(S^n)$ must be the zero map.

☐

Exercise: Show that there are maps $T^n \to S^n$ of non-zero degree.

Proof of Proposition 6.1. This is a variant of the proof of homotopy invariance, but instead using a "twisted prism" operator.

Let $\varepsilon_n := (-1)^{\frac{n(n+1)}{2}}$, and let $\rho : C_n(X) \to C_n(X)$ be defined by:

$$[v_0, \dots, v_n] \longmapsto \varepsilon_n[v_n, \dots, v_0].$$

Note that ε is just the sign of the corresponding permutation, $(0,1,\dots,n) \mapsto (n,\dots,1,0)$.

The key claim is that ρ is a chain map, which is chain homotopic to the identity. Given this, we have:

$$\begin{aligned}(\rho^*\varphi)\cdot((\rho^*\psi)[\sigma]) &= \varphi\left(\varepsilon_k\sigma|_{[v_k,\dots,v_0]}\right)\cdot\psi\left(\varepsilon_l\sigma|_{[v_{k+l},\dots,v_k]}\right)\\ &= \varepsilon_k\varepsilon_l\varphi\left(\sigma|_{[v_k,\dots,v_0]}\right)\cdot\psi\left(\sigma|_{[v_{k+l},\dots,v_k]}\right)\end{aligned}$$

and

$$\rho^*(\psi\cdot\varphi)[\sigma] = \varepsilon_{k+l}\psi\left(\sigma|_{[v_{k+l},\dots,v_k]}\right)\cdot\varphi\left(\sigma|_{[v_k,\dots,v_0]}\right)$$

and thus on comparing the RHS's, we see

$$\varepsilon_{k+l}\rho^*(\psi\cdot\varphi) = \varepsilon_k\varepsilon_l\rho^*\varphi\cdot\rho^*\psi.$$

But then since $\varepsilon_{k+l} = (-1)^{kl}\varepsilon_k\cdot\varepsilon_l$, and so since ρ is the identity on H^* (since it is chain homotopic to the identity and so descends to the same map on cohomology) we get that

$$\varphi\cdot\psi = (-1)^{kl}\psi\cdot\varphi$$

on H^*, as required.

So it remains to show the 'key claim' above. So let $\sigma \in C_n(X)$. Then we can compute

$$d(\rho(\sigma)) = \varepsilon_n\cdot\left(\sum_i(-1)^i\sigma|_{[v_n,\dots,\hat{v}_{n-i},\dots,v_0]}\right)$$

and

$$\begin{aligned}\rho(d(\sigma)) &= \rho\left(\sum_i(-1)^i\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right)\\ &= \varepsilon_{n-1}\sum_i(-1)^{n-i}\sigma|_{[v_n,\dots,\hat{v}_{n-i},\dots,v_0]}\end{aligned}$$

where we have relabelled the indices ($i \mapsto n-i$). Note $\varepsilon_n = (-1)^n\varepsilon_{n-1}$.

$$\begin{tikzcd}
C_n(X) \arrow[r, "d"] \arrow[d, "\rho"] & C_{n-1}(X) \arrow[d, "\rho"] \\
C_n(X) \arrow[r, "d"] & C_{n-1}(X)
\end{tikzcd}$$

To see ρ is chain homotopic to the identity, we want $P : C_n(X) \to C_{n+1}(X)$ such that

$$dP + Pd = \rho - \text{id}.$$

So let $\pi : \Delta^n \times [0,1] \to \Delta$ be the projection. Then for $\sigma = [v_0,\dots,v_n]$ define P, the so-called **twisted prism operator**, by

$$P(\sigma) := \sum_i(-1)^i\varepsilon_{n-i}(\sigma\circ\pi)[v_0,\dots,v_i,w_n,\dots,w_i].$$

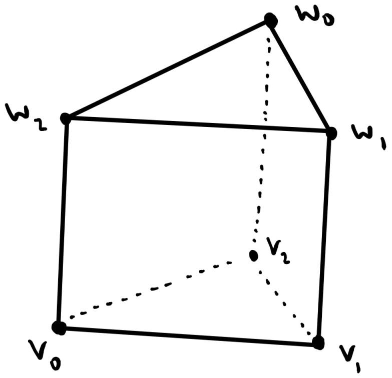


FIGURE 20. An illustration of the twisted prism operator.

Then one can compute:

$$d(P(\sigma)) = \sum_{j\leq i}(-1)^i(-1)^j\varepsilon_{n-i}[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_n,\ldots,w_i] + \sum_{j\geq i}(-1)^i(-1)^{i+1+n-j}\varepsilon_{n-i}[v_0,\ldots,v_i,w_n,\ldots,\hat{w}_j,\ldots,w_n]$$

and

$$P(d\sigma) = \sum_{i<j}(-1)^i(-1)^j\varepsilon_{n-i-1}[v_0,\ldots,v_i,w_n,\ldots,\hat{w}_j,\ldots,w_i] + \sum_{i>j}(-1)^{i-1}(-1)^j\varepsilon_{n-i}[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_n,\ldots,w_i].$$

In the first sum we group the $i=j$ terms, and these give:

$$\varepsilon_n[w_n,\ldots,w_0]-[v_0,\ldots,v_n] \underbrace{+\sum_{i>0}\varepsilon_{n-i}[v_0,\ldots,v_{i-1},w_n,\ldots,w_i]+\sum_{i<n}(-1)^{n+i+1}\varepsilon_{n-i}[v_0,\ldots,v_i,w_n,\ldots,w_{i+1}]}_{\text{these sums cancel under reindexing } i\mapsto i-1}.$$

All the remaining terms of the $d(P(\sigma))$ sum match terms in $P(d\sigma)$, again using $\varepsilon_{n-i}=(-1)^{n-i}\varepsilon_{n-i-1}$. Then we can see the result and so we are done. [All the sign matching isn't very transparent, so need to check it.]

$\square$

Digression: Whilst the above argument may seem rather random, we give a sketch overview of a more conceptual proof, which led to the above as (we gave the above proof as it is more elementary).

Let $\Delta : C_{k+l}(X)\to C_k(X)\otimes C_l(X)$ be:

$$[v_0,\ldots,v_{k+l}]\longmapsto[v_0,\ldots,v_k]\otimes[v_k,\ldots,v_{k+l}]$$

and $\tilde{\Delta} : C_{k+l}(X)\to C_k(X)\otimes C_l(X)$ be

$$[v_0,\ldots,v_{k+l}]\longmapsto(-1)^{kl}[v_l,\ldots,v_{l+k}]\otimes[v_0,\ldots,v_l].$$

For $\varphi \in C^k$ and $\psi \in C^l$, we have

$$\varphi \cdot \psi = \text{mult}_{\mathbb{Z}} \circ (\varphi \otimes \psi) \circ \Delta \quad \text{and} \quad (-1)^{kl} \psi \circ \varphi = \text{mult}_{\mathbb{Z}} \circ (\varphi \otimes \psi) \circ \tilde{\Delta}$$

where $\text{mult}_{\mathbb{Z}}$ is just usual multiplication, i.e. $\text{mult}_{\mathbb{Z}}(a,b) = ab$, for $a, b \in \mathbb{Z}$.

So commutativity of the cup product is really the statement that Δ and $\tilde{\Delta}$ agree (suitably interpreted), i.e. $\exists!$ natural chain map $C_*(X) \to C_*(X) \otimes C_*(X)$. If the map is to be natural in spaces, it suffices to construct it for Δ, i.e. to show that there is a unique natural chain map

$$C_*(\Delta^n) \to C_*(\Delta^n) \otimes C_*(\Delta^n)$$

for each n. The complex $C_*(\Delta^n)$ <u>and</u> $C_*(\Delta^n) \otimes C_*(\Delta^n)$ are both **free resolutions** of $\mathbb{Z}$ (in degree 0).

"Acyclic models" (see, e.g. Spanier's book) says $\mathbb{Z}$ has a unique free resolution up to chain homotopy equivalence. We can apply this to therefore show that Δ and $\tilde{\Delta}$ are chain homotopy equivalent, and naturality then gives it for all spaces, as we wanted.

6.1. **Critical Points.**

If X is a finite cell complex, $H^*(X)$ is a finitely generated ring. So we can introduce new invariants of spaces, e.g. the minimal number of ring generators, or the **cup length**:

Definition 6.2. *The **cup length** of a finite cell complex X is:*

$$cl(X) := \max\left\{N : \exists \alpha_i \in H^{>0}(X) \text{ for } 1 \le i \le N \text{ such that } \alpha_1 \smile \cdots \smile \alpha_N \neq 0 \in H^*(X)\right\}$$

i.e. the length of the biggest product which is non-zero.

Example 6.2. *For $n > 0$, we have $cl(S^n) = 1$, whilst for $n > 0$ we have $cl(T^n) = n$.*

Now let X be a space which admits an open cover $\{U_\alpha\}_{\alpha \in A}$ by sets such that $U_\alpha \hookrightarrow X$ (inclusion map) are homotopic to constant maps (sp $U_\alpha \simeq \{\text{point}\}$).

Definition 6.3. *The **category** $\nu(A)$ of a subspace $A \subset X$ (X as above) is the least $N \in \mathbb{N} \cup \{\infty\}$ such that A can be covered by N open sets U_i such that the inclusion map $U_i \hookrightarrow$ is homotopic to a constant map.*

Note: If M is a <u>closed</u> manifold (so compact without boundary), then M has finite category.

Example 6.3. *By taking U_1, U_2 to be hemispheres, we have $\nu(S^n) = 2$ for all n.*

The map $\nu : \{\text{Subsets of } X\} \to \mathbb{N} \cup \{\infty\}$ is easily check to satisfy:

(i) If $A \subset X$, then $\exists$ some U open with $A \subset U \subset X$ and $\nu(A) = \nu(U)$.
(ii) If $A \subset B$ then $\nu(A) \le \nu(B)$.

(iii) $\nu(A \cup B) \leq \nu(A) + \nu(B)$.

(iv) $\nu(\emptyset) = 0$ and $\nu(\{\text{point}\}) = 1$.

(v) ν is a homeomorphism invariant, so if $\varphi : X \to X'$ is a homeomorphism, then $\nu(A) = \nu(\varphi(A))$ for all A.

From Example sheet 2, we also know that $\mathrm{cl}(X) < \nu(X)$.

Theorem 6.2. *Let M be a closed, smooth, connected manifold. Then any smooth function $f : M \to \mathbb{R}$ has at least $1 + cl(M)$ critical points.*

Example 6.4. *Any smooth map $f : T^n \to \mathbb{R}$ has at least $(n+1)$—critical points.*

Proof of Theorem 6.2 (Non-Examinable). We will use some basic differential topology. So let $f : M \to \mathbb{R}$ be smooth, and for $c \in \mathbb{R}$ define $M^c := f^{-1}((-\infty, c])$.

Pick a Riemannian metric g on M, and hence a vector field ∇f, the gradient of f(iv). Then let $\varphi^t : M \to M$ be the flow of the vector field $-\nabla f$. Set $c_j := \sup\{c : \nu(M^c) < j\}$. Note that $c_1 = \min f$ and $c_{\nu(M)} = \max f$.

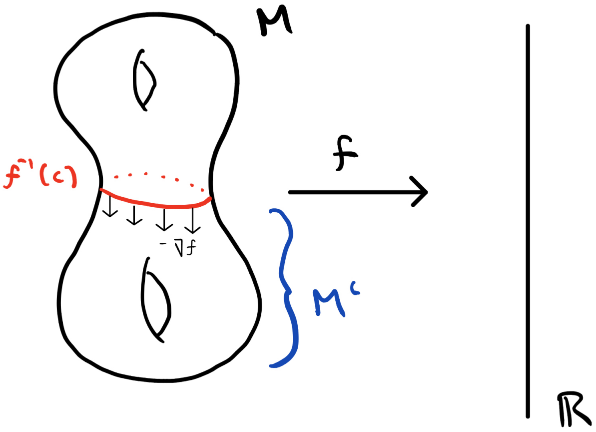


FIGURE 21. An illustration of the setup.

The key input from differential topology is:

(iv) **Aside:** g is a non-degenerate symmetric bilinear form on T_xM which is smooth in x. Hence g induces an isomorphism $T_xM \xrightarrow{\cong} T_x^*M$ via $v \mapsto g(v, \cdot) \equiv \langle v, \cdot \rangle_g$. Then ∇ is the vector field associated to $\mathrm{d}f \in \Gamma(T^*M)$, i.e. it has the property that

$$\left\langle \nabla f, \frac{\mathrm{d}c}{\mathrm{d}t} \right\rangle_g = \frac{\mathrm{d}(f \circ c)}{\mathrm{d}t}$$

for any smooth curve $c : I \to M$.

> **Claim:** If $c \in \mathbb{R}\backslash\text{Critical}(f)$, then $\exists t > 0$ and $\delta > 0$ such that $\varphi^t\left(M^{c+\delta}\right) \subset M^{c-\delta}$.
>
> [Note: $\nabla f_x = 0 \iff x \in \text{Crit}(f)$.]

We will take this result for granted. Thus we see that $c_j \in \text{Critical}(f)$ for all j, since ν is a homeomorphism invariant and φ^t is a flow by diffeomorphisms.

> **Claim:** Either $c_j < c_{j+1}$, <u>or</u> $f^{-1}(c_j)$ contains infinitely many critical points of f.
>
> [If so, f has at least $\nu(M)$ critical points, and we know $\nu(M) > \text{cl}(M)$, and so we will be done.]

So suppose $f^{-1}(c_j)$ contains only finitely many critical points. Then if $S \subset M$ is a <u>finite</u> set in a connected manifold, $\exists$ an open disc U containing S. Then, $\nu(S) \leq \nu(U) = 1$, and $\nu(S) = 1$. Now,

$$\begin{aligned}\nu\left(M^{c_j+\delta}\right) &\leq \nu\left(M^{c_j+\delta}\backslash U\right) + 1 && \text{and the flow } \varphi^t \text{ for small time } t\\ &\leq \nu\left(M^{c_j-\delta}\right) + 1 && \text{pushes } M^{c_j+\delta}\backslash U \text{ into } M^{c_j-\delta}\\ &< j+1.\end{aligned}$$

So $c_{j+1} \geq c_j + \delta > c_j$, and so we always have $c_{j+1} > c_j$. Hence we are done.

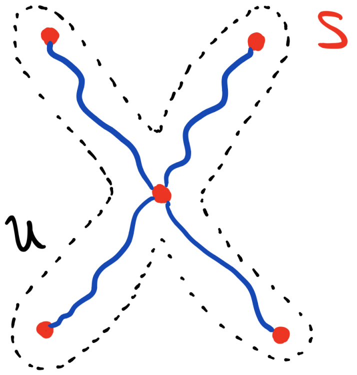


FIGURE 22. An illustration of the set U. The finite set S is shown in red, with paths joining all points of S together. Then we find a neighbourhood of each of these paths in M to find U.

$\square$

Remark: Morse theory (see also *Floer Theory*) is all about recovering $H_*(M)$ from the critical points of smooth functions on the manifold.

7. VECTOR BUNDLES

Let B be a space.

Definition 7.1. *A **vector bundle** is a pair (E,p), where $p : E \to B$ and we have*

(i) $E = \amalg_{b\in B} E_b$ for a family of vector spaces $(E_b)_{b\in B}$ of some fixed dimension n

(ii) a topology on E such that $p : E \to B$ is continuous

(iii) for all $b \in B$, $\exists$ an open set $U \ni b$ and a homeomorphism $\psi_U : E|_U \to U \times \mathbb{R}^n$, where $E|_U := p^{-1}(U)$, such that

$$\begin{tikzcd} E|_U \arrow[rr, "\psi_U"] \arrow[dr, "p"'] & & U \times \mathbb{R}^n \arrow[dl, "projection"] \\ & U & \end{tikzcd}$$

*commutes, and for all $y \in U$, $\psi_U|_{p^{-1}(y)} : E_y \xrightarrow{\cong} \{y\} \times \mathbb{R}^n$ is a linear isomorphism ($E_y = p^{-1}(y)$). This condition says that the vector bundle is **locally trivial**, and the collection of open sets U used is called a **trivialising cover**.*

Notation. E is called the **total space**, B is the **base space**, and n is called the **rank** of E. Each $E_y := p^{-1}(y)$ is called the **fibre** of the bundle at $y \in B$.

Definition 7.2. *A map $s : B \to E$ such that $p \circ s = \mathrm{id}_B$ is called a **section** of the vector bundle E.*

Note: There is a canonical zero-section, assigning to each $b \in B$ the point $0 \in E_b$. Note that the zero section $0_s : B \hookrightarrow E$ is a deformation retract, with homotopy inverse p (the vector bundle map).

Example 7.1. *For any space B, there is the **trivial vector bundle** $p : E \to B$, where $E = B \times \mathbb{R}^n$ and $p : E \to B$ is the projection $(b,v) \mapsto b$.*

We then have some basic operations on vector bundles to form new ones from old ones.

(i) **Pullback bundle.** Suppose $E \xrightarrow{p} X$ is a vector bundle and $f : Y \to X$ is any map. Then we can form the **pullback bundle** f^*E over Y by:

$$f^*E := \{(e,y) \in E \times Y : p(e) = f(y)\}.$$

So, $(f^*E)_y := E_{f(y)}$.

(ii) **Whitney Sum.** Given two vector bundles over X, $E \xrightarrow{p} X$ and $F \xrightarrow{q} X$, we can form $E \oplus F \to X$ via:

$$E \oplus F := \{(e,f) \in E \times F : p(e) = q(f)\} \equiv E \times_x F.$$

So, $(E \oplus F)_x = E_x \oplus F_x$.

Note: Pullback and Whitney sum in particular take trivial bundles to trivial bundles, and commute with restriction to open sets in the base, i.e. if $U \subset X$ is open, then

$$f^*(E|_U) = (f^*E)|_{f^{-1}(U)} \quad \text{and} \quad (E \oplus F)|_U = E|_U \oplus F|_U.$$

So the pullback and Whitney sum are also locally trivial [**Exercise** to check details].

Now suppose $B = \bigcup_{\alpha \in A} U_\alpha$ is a trivialising cover for $E \xrightarrow{p} B$. Then by definition we have maps $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ for each α. Then we have a diagram:

$$\begin{tikzcd} E|_{U_\alpha \cap U_\beta} \arrow[r, "\psi_\alpha"] \arrow[d, "\psi_\beta"] & U_\alpha \times \mathbb{R}^n \arrow[dl, dashed] \\ U_\beta \times \mathbb{R}^n & \end{tikzcd}$$

i.e. for each point in $p \in U_\alpha \cap U_\beta$, the composition $\psi_\beta \circ \psi_\alpha^{-1}|_{\{p\}}$ gives an isomorphism $\{p\} \times \mathbb{R}^n \to \{p\} \times \mathbb{R}^n$. Thus we get a map

$$\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$$

which sends a point to the above isomorphism. These maps $(\psi_{\beta\alpha})_{\alpha,\beta \in A}$ can then be shown to satisfy

$$\begin{cases} \psi_{\alpha\alpha} = \mathrm{id} \\ \psi_{\alpha\beta} \circ \psi_{\beta\gamma} \circ \psi_{\gamma\alpha} = \mathrm{id}. \end{cases}$$

These conditions are known as the **cocycle conditions**. We can then reconstruct the vector bundle E from the cocycle data via:

$$(\amalg_{\alpha \in A}(U_\alpha \times \mathbb{R}^n)) / \sim$$

where $(x, v) \sim (x, \psi_{\beta\alpha}(x)(v))$ for $x \in U_\alpha \cap U_\beta$.

So a vector bundle is determined by a trivialising open cover and maps $\{\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})\}$ satisfying the cocycle conditions. This data for the vector bundle is known as the **cocycle data**.

Example 7.2. *If $p : E \to B$, $q : F \to B$ are vector bundles and $(U_\alpha)_{\alpha \in A}$ is trivialising for both, then we can define the* ***tensor product bundle****, $E \otimes F$, to be the vector bundle associated to the matrix-valued functions (defining cocycle data):*

$$\psi^E_{\beta\alpha} \otimes \psi^F_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_{n(E)n(F)}(\mathbb{R})$$

for E of rank $n(E)$ and F of rank $n(F)$.

Example 7.3. *Suppose M is a smooth manifold. Then by definition it has a smooth atlas. M is covered by sets U_α and $\exists \varphi_\alpha : U_\alpha \xrightarrow{\cong} D^n \subset \mathbb{R}^n$ and the* ***transition functions*** *$\varphi_{\beta\alpha} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are smooth.*

Then we define the ***tangent bundle*** *of M, denoted TM, to be the vector bundle associated to the transition matrices $\psi_{\beta\alpha} := Jac(\varphi_{\beta\alpha})$. The chain rule implies that these satisfy the cocycle condition, and so the tangent bundle is the vector bundle associated to these cocycles.*

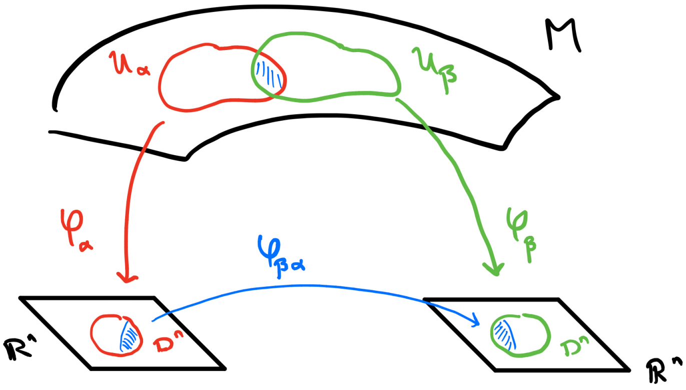


FIGURE 23. An illustration of manifold charts and transition maps $\varphi_{\beta\alpha}$.

Definition 7.3. *Let $X = Gr_k(\mathbb{R}^n) := \{k-dimensional\ subspaces\ of\ \mathbb{R}^n\}$ be the* ***Grassmannians****. Then clearly $GL_n(\mathbb{R})$ acts transitively on X, and so we can topologise X as its quotient space, and in fact:*

$$X = \frac{O(n)}{O(k)\times O(n-k)}.$$

The ***tautological bundle*** *$E \to X$, denoted E_{taut}, is defined as follows: its fibre at $x \in Gr_k(\mathbb{R}^n)$ is the corresponding k—dimensional subspace of $\mathbb{R}^n$ (that is, itself), i.e. $E_x = \langle x\rangle \cong \mathbb{R}^k \subset \mathbb{R}^n$.*

Note that the tautological bundle is naturally a subspace of the trivial bundle $X \times \mathbb{R}^n$.

Of course we must actually check that the tautological bundle is a vector bundle. In particular, we need to check that E is locally trivial.

Proof that the tautological bundle is locally trivial. Pick an inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$. Given $x \in X$, let

$$U_x := \{y \in X : E_y \cap E_x^\perp = \{0\}\}.$$

Then define $E|_{U_x} \to U_x \times E_x = U_x \times \mathbb{R}^k$ by: $(y,\xi) \mapsto \big(y, \text{pr}_{\langle x\rangle}(\xi)\big)$, where $\text{pr}_{\langle x\rangle}$ is the orthogonal projection onto $\langle x\rangle$(v). Then this is [**Exercise** to check] an isomorphism $E_y \to E_x$ for all $y \in U_x$, by definition of U_x.

☐

Remark: There is an obvious notion of a **complex vector bundle**: we require that all $E_x \cong \mathbb{C}^n$, and that the local trivialisations are $\mathbb{C}$-linear isomorphisms on fibres. So we also get in the same as above that ∃ a tautological bundle

$$E \to \text{Gr}_k(\mathbb{C}^n) \equiv \frac{U(n)}{U(k)\times U(n-k)}$$

and in particular ∃ a tautological <u>line</u> bundle $\mathscr{L} \to \mathbb{C}P^n$, or $\mathscr{L} \to \mathbb{R}P^n$.

(v) Note that even though $x \in \text{Gr}_k(\mathbb{R}^n)$ is a k—dimensional subspace, we still denote $\langle x\rangle$ by this subspace.

Note: A **line bundle** is a vector bundle of rank 1. Note that this notion depends on your base field $\mathbb{R}$ or $\mathbb{C}$, as a $\mathbb{C}$-line bundle is a rank 2 $\mathbb{R}$-bundle, etc.

Definition 7.4. *If $E \to X$ is a vector bundle, a subspace $F \subset E$ is a* ***vector subbundle*** *of E if for all $x \in X$, we have a vector subspace $F_x \subset E_x$ with $F = \bigcup_{x\in X} F_x \subset E$ (disjoint union), and $\exists$ a locally trivialising cover $(U_\alpha)_\alpha$ for E such that:*

$$\begin{tikzcd} E|_{U_\alpha} \arrow[r] & U_\alpha \times \mathbb{R}^n \\ F|_{U_\alpha} \arrow[r] \arrow[u, hook] & U_\alpha \times \mathbb{R}^k \arrow[u, hook] \end{tikzcd}$$

commutes for some fixed k, where the vertical maps are inclusions whilst the top map is from the trivialising cover.

Note that when we say "$F \subset E$" is a subspace of the vector bundle E, we mean $F_x \subset E_x$ is a subspace for each x.

Given a vector subbundle $F \subset E$, there is a **quotient bundle** E/F with fibre E_x/F_x at x.

Definition 7.5. *Let X be a compact Hausdorff space. Let $(U_\alpha)_\alpha$ be an open cover of X. Then a* ***partition of unity (subordinate to the cover)*** *is a collection of maps $(\lambda_\alpha)_\alpha$, $\lambda_\alpha : X \to \mathbb{R}_{\geq 0}$ such that*

- *(i) $supp(\lambda_\alpha) \subset U_\alpha$.*
- *(ii) For all $x \in X$, $|\{\alpha : x \in supp(\lambda_\alpha)\}| < \infty$, i.e. only finitely many λ_α are non-zero at x.*
- *(iii) For all $x \in X$, $\sum_{\alpha\in A} \lambda_\alpha(x) = 1$.*

Such a space where these always exist is called ***paracompact****.*

Lemma 7.1. *Compact Hausdorff spaces are paracompact.*

Proof. None given. □

Lemma 7.2. *Let X be a compact Hausdorff space. Then if $E \to X$ is a vector bundle, then E admits an inner product, i.e. $\exists \lambda : E \otimes E \to \mathbb{R}$ such that for all $x \in X$, $\lambda|_{E_x \otimes E_x} : E_x \otimes E_x \to \mathbb{R}$ is a non-degenerate inner product on E_x.*

Moreover for all $x \in X$ and $\xi_x \in E_x$, $\exists$ a section $s : X \to E$ with $s(x) = \xi_x$.

Proof. Fix an inner product, which we shall denote by $\langle\cdot,\cdot\rangle_\alpha$, on $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$, where $(U_\alpha)_\alpha$ is a trivialising open cover for E. Then take a partition of unity $(\lambda_\alpha)_\alpha$ subordinate to this cover. Then set:

$$\langle \xi_x, \eta_x \rangle := \sum_{\alpha\in A} \langle \xi_x, \eta_x \rangle_\alpha \cdot \lambda_\alpha(x) \qquad \text{if } \xi_x, \eta_x \in E_x.$$

Note that this makes sense, since even if $x \notin U_\alpha$ (which would mean $\langle \xi_x, \eta_x \rangle_\alpha$ is not defined), then $\lambda_\alpha(x) = 0$ and so we can just call this term in the sum 0.

It is then easy to check that this is well-defined (i.e. always finite) and is an inner product [**Exercises** to check]. This proves the first claim.

Similarly for the second statement, take a constant section $s_\alpha(x) \equiv \xi_x$ for $x \in U_\alpha$, which is a section of $E|_{U_\alpha} = U_\alpha \times \mathbb{R}^n$, and then extend this to a global section using the partition of unity. [**Exercise** to check.]

□

Corollary 7.1. *Let X be compact Hausdorff and let $E \to X$ be a vector bundle of rank n. Then $\exists N > n$ and a continuous map $f : X \to \mathrm{Gr}_n(\mathbb{R}^N)$ such that $E \cong f^*E_{\text{taut}}$ is the pullback of the tautological bundle.*

Remark: A (structured, guided, reasonable) question on Example Sheet 3 shows:

$$\mathrm{Vect}_k(X)/\cong \;\longleftrightarrow\; [X, \mathrm{Gr}_n(\mathbb{R}^\infty)]$$

are in 1:1 correspondance, where $\mathrm{Vect}_k/\cong$ is the set of rank k vector bundles up to isomorphism, and the RHS is the set of homotopy classes of maps X to $\bigcup_{k\geq 0} \mathrm{Gr}_n(\mathbb{R}^k)$.

Thus it would seem reasonable to think that understanding the cohomology of Grassmannians would help us understand the cohomology of vector bundles and thus manifolds.

Proof of Corollary 7.1. Since sections through any point of E exists (by Lemma 7.2), using compactness of X we can find finitely many sections $s_1, \ldots, s_N$ of E such that span $\langle s_i(x) : 1 \leq i \leq N \rangle = E_x$, i.e. they $\underline{\text{span}}$ at $\underline{\text{every}}$ point (not a basis, just span. So we get local span, then use compactness to get this globally, etc). This defines a map $E : X \to \mathbb{R}^N$ via:

$$(x, \xi) \longmapsto (x, (\langle s_1(x), \xi \rangle, \ldots, \langle s_N(x), \xi \rangle))$$

where $\langle\cdot,\cdot\rangle$ is an inner product on E (exists by Lemma 7.2). Then since the $(s_i)_i$ span, this $\underline{\text{embeds}}$ E into $X \times \mathbb{R}^N$ as a subbundle of the trivial bundle [**Exercise** to check].

Now we define $f : X \to \mathrm{Gr}_n(\mathbb{R}^N)$ by $x \mapsto [E_x] \subset \mathbb{R}^N$. Then $E = f^*E_{\text{taut}}$ and so we are done.

□

Remark: We have also shown (under the same hypotheses on X) that given E, $\exists$ another bundle $F \to X$ such that $E \oplus F \cong \mathbb{R}^N \times X$ is a trivial bundle: just take $F = E^\perp$ for $E \subset \mathbb{R}^N \times X$ and for a fixed inner product on $\mathbb{R}^N$.

7.1. **The Thom Isomorphism.**

Definition 7.6. *Let $p : E \to X$ be a vector bundle of rank n. Then we say that E is* ***oriented*** *if $\forall x \in X$, we have a distinguished generator of $H^n(E_x, E_x \backslash \{0\})^{\text{(vi)}}$. Call this generator ε_x. Then we require that $x \mapsto \varepsilon_x$ should vary locally trivially, in the sense that if $x \in U \subset X$ is a trivialising open set, then:*

$$\begin{tikzcd}
E|_U \arrow[r, "\cong"] & U \times E_x \\
E_y \arrow[u, hook] \arrow[r, "\cong"] & \{y\} \times E_x \arrow[u, hook]
\end{tikzcd}$$

the induced map on cohomology should send $\varepsilon_x \mapsto \varepsilon_y$.

Notation: We write $E^\# := E\backslash\{\text{zero-section}\}$, pronounced "$E$ sharp". So $E_x^\# = E_x \backslash \{0\}$.

Example 7.4. *Complex vector bundles are always canonically oriented (see Example Sheet 3).*

Example 7.5. *If $E \to X$ is defined via transition matrices/cocycle data $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$, with $(U_\alpha)_\alpha$ a trivialising cover, then E is orientable if $\psi_{\beta\alpha}$ has image in $GL_n^+(\mathbb{R})$, the subgroup of matrices of positive determinant, for all α, β (see Example Sheet 3).*

Note: If M is a manifold, it has a tangent bundle TM. We can see that TM being orientable in our sense above is equivalent to TM being orientable in any other reasonable sense we may have seen previously. (e.g. as in Example 7.5).

Theorem 7.1 (Thom Isomorphism Theorem). *Let $E \to X$ be an oriented vector bundle of rank n. Then:*

- *(i)* $H^k(E, E^\#) = 0$ *if* $k < n$,
- *(ii)* $\exists\, !\ u_E \in H^n(E, E^\#)$ *such that* $u_E|_{E_x} = \varepsilon_x$ *for all* $x \in X^{\text{(vii)}}$,
- *(iii) The map* $H^i(X) \to H^{i+n}(E, E^\#)$ *sending* $\alpha :\mapsto \pi^*(\alpha) \cdot u_E$, *is an isomorphism for all* i.

Definition 7.7. u_E *(this canonical cohomology class) is called the* ***Thom class*** *of the bundle* E.

Under the map $H^n(E, E^\#) \to H^n(E) \xrightarrow{\cong} H^n(X)$ (the later map induced by the inclusion $X \hookrightarrow E$ via the zero-section) the Thom class $u_E \mapsto e_E$ maps to the **Euler class** of E, $e_E \in H^{\text{rank}(E)}(X)$. Thus we can think of the Euler class as the restriction of the Thom class to the 0-section.

(vi) Note that $H^n(E_x, E_x \backslash \{0\}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) \cong H^{n-1}(S^{n-1}) \cong \mathbb{Z}$, so it makes sense to talk about a generator.

(vii) Here, | denotes the restriction, i.e. the pullback under the natural inclusion $(E_x, E_x^\#) \hookrightarrow (E, E^\#)$.

Remark: If $E \to X$ is oriented and $f : Y \to X$, the pullback bundle $f^*E \to Y$ inherits a canonical orientation via:

$$(f^*E)_y \xrightarrow{\cong} E_{f(y)}.$$

The uniqueness part of the Thom isomorphism theorem then says that:

$$u_{f^*E} = \hat{f}^*(u_E)$$

if $\hat{f} : \big(f^*E, (f^*E)^\#\big) \to (E, E^\#)$. Hence we see $e_{f^*E} = f^*e_E \in H^n(Y)$.

Definition 7.8. *A rule $E \mapsto c(E)$ that assigns to a bundle $E \to X$ (perhaps satisfying some conditions, e.g. orientability) a cohomology class $c(E) \in H^*(X)$ such that $c(f^*E) = f^*c(E)$ for all $f : Y \to X$ is called a* ***characteristic class*** *of these types of vector bundle.*

Example 7.6. *The Euler class is a characteristic class of oriented vector bundles.*

Proof of Thom Isomorphism. We will prove this under the simplifying hypothesis that X admits a finite trivialising cover for E (the general result then follows from Zorn's lemma).

To prove this, we induct on the number of sets in such a finite trivialising cover.

Base case. For one such set, we have $E = X \times \mathbb{R}^n$ is a trivial vector bundle here. Then we fix $\varepsilon \in H^n(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$. Then the Künneth theorem and the 5-Lemma give that, if $H^*(Y)$, $H^*(Z)$ and $H^*(Y,Z)$ are finitely generated and free, then

$$H^*(X \times Y, X \times Z) \cong H^*(X) \otimes H^*(Y,Z).^{\text{(viii)}}$$

So take $(Y,Z) = (\mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$. This shows that:

$$H^*(E,E^\#) \cong H^*(X) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) \ni 1 \otimes \varepsilon =: u_E \in H^n(E,E^\#).$$

Then all of (i)—(iii) of the theorem are immediate.

Inductive step. Assume that the result is known for oriented bundles trivialised over a cover of $< k$ open sets. Now suppose E is trivialised in a cover of X with k open sets. So we can write $X = U \cup V$, where U, V are such that the theorem is known for $E|_U$, $E|_V$ and $E|_{U\cap V}$ (i.e. if U is just one of the k sets and V is the other $k-1$).

Then consider Mayer-Vietoris (we will check that the MV sequence for pairs holds shortly):

$$\begin{tikzcd}
H^{i-1}(E|_{U\cap V}, E^\#|_{U\cap V}) \arrow[r] & H^i(E,E^\#) \arrow[r] & H^i(E|_U, E^\#|_U) \oplus H^i(E|_V, E^\#|_V) \arrow[dll, rounded corners] \\
H^i(E|_{U\cap V}, E^\#|_{U\cap V}) \arrow[r] & H^{i+1}(E,E^\#) \arrow[r] & \cdots
\end{tikzcd}$$

For $i < n$ this gives:

$$0 \longrightarrow H^i(E,E^\#) \longrightarrow 0$$

and so indeed (i) holds by induction.

(viii) This turns out to be non-obvious. We shall discuss it after the proof.

If $i = n$ we get:

$$0 \longrightarrow H^n(E,E^{\#}) \xrightarrow{\varphi} \underset{\ni u_{E|_U}}{H^n(E|_U,E^{\#}|_U)} \oplus \underset{\ni u_{E|_V}}{H^n(E|_V,E^{\#}|_V)} \longrightarrow \underset{u_{E|_{U\cap V}}}{H^n(E|_{U\cap V},E^{\#}|_{U\cap V})} \longrightarrow \cdots$$

where the red elements exist by induction.

The uniqueness part of the theorem for the Thom class of $E_{U\cap V}$ shows that $u_{E|_U}$ and $u_{E|_V}$ vitg restrict to $u_{E|_{U\cap V}}$ on $U\cap V$.

So $\exists$ an element $u_E \in H^n(E,E^{\#})$ such that $\varphi(u_E) = (u_{E|_U}, u_{E|_V})$. Now φ is injective (by exactness of the sequence), and so u_E is uniqueness. This proves (ii) of the theorem.

For the last part, to see $T : \alpha \mapsto (\pi^*\alpha)\cdot u_E$ is an isomorphism (here $\cdot$ is the cup product), we have the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(E,E^{\#}) \arrow[r] & H^i(E|_U,E^{\#}|_U)\oplus H^i(E|_V,E^{\#}|_V) \arrow[r] & H^i(E|_{U\cap V},E^{\#}|_{U\cap V}) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{i-n}(X) \arrow[r] \arrow[u, "T"] & H^{i-n}(U)\oplus H^i(V) \arrow[r] \arrow[u, "T", "\cong"'] & H^{i-n}(U\cap V) \arrow[r] \arrow[u, "T", "\cong"'] & \cdots
\end{tikzcd}$$

where the red isomorphisms are known by induction. So if we can just show that all the squares commute, then the 5-Lemma will imply that T is an isomorphism for the bundle $E \to X$, and so by induction we would be done.

So we need to check:

$$\begin{tikzcd}
H^i(E|_{U\cap V},E^{\#}|_{U\cap V}) \arrow[r, "\partial^*_{MV}"] & H^{i+1}(E,E^{\#}) \\
H^{i-n}(U\cap V) \arrow[r, "\partial^*_{MV}"] \arrow[u, "T"] & H^{i-n+1}(X) \arrow[u, "T"]
\end{tikzcd}$$

commutes.

So let $\varphi \in C^n(E,E^*)$ be a cocycle representing u_E. Then $\varphi|_{E|_U}$ represents $u_{E|_U}$. So take $[\alpha] \in H^{i-n}(U\cap V)$. To define $\partial^*_{MV}(\alpha)$, write

$$\alpha = \psi_U - \psi_V$$

for $\psi_U \in C^{i-n}(U)$ and $\psi_V \in C^{i-n}(V)$, and set:

$$[\partial^*_{MV}(\alpha)] := [d^*\psi_U]$$

for $d^* : C^{i-n}(U) \to C^{i-n+1}(U)$.

So for the red direction in the diagram, we have this map is: $\alpha \mapsto \pi^*(d^*\psi_U)\cdot\varphi$. But $\pi^*(\alpha)u_{E|_{U\cap V}}$ is a difference of cochains $\pi^*\psi_U \cdot \varphi_{E|_U} - \pi^*\psi_V \cdot \varphi|_{E_V}$, which lie in $C^i(E|_U,E^{\#}|_U)$ and $C^i(E|_V,E^{\#}|_V)$.

Then for the green direction on the diagram, this ends $\alpha \mapsto d^*(\pi^*(\psi_U)\cdot\varphi_{E|_U})$.

Now since $\pi^*\circ d^* = d^*\circ\pi^*$ and also $d^*\varphi = 0$ since φ was a cocycle, and so we see the two expressions above for going round the diagram in two different ways agree, and thus the diagram commutes. Hence we are done as explained before.

$\square$

To truly finish the proof of the Thom isomorphism, we need to prove two results which we claimed within the proof. The first of these we in the base case (footnote (viii)) whilst the second was the MV sequence for pairs.

Remark: (Relative Version of Mayer-Vietoris).

Recall that $C_*(A+B)$ was defined (for $X = A\cup B$ such that the interiors of A, B covered X) by the subcomplex of simplicies lying wholly in A <u>or</u> B. The small simplicies theorem then said that $C_*(A+B)\to C_*(X)$ is an isomorphism on homology. Dually, $C^*(X)\to C^*(A+B) =: \mathrm{Hom}(C_*(A+B);\mathbb{Z})$ is an isomorphism on cohomology.

Then $\exists$ a s.e.s of cochain complexes defined via:

$$\begin{tikzcd} 0 \arrow[r] & C^*(A+B) \arrow[r, "R"] & C^*(A)\oplus C^*(B) \arrow[r, "S"] & C^*(A\cap B) \arrow[r] & 0 \end{tikzcd}$$

where $R: \psi \mapsto (\psi|_A, \psi|_B)$ and $S: (u,v)\mapsto u|_{A\cap B} - v|_{A\cap B}$, which gives the usual MV sequence.

Now suppose $(X,Y) = (A\cup B, C\cup D)$, where $C\subset A$ and $D\subset B$, with the interiors of C and D covering Y. Then we have:

$$\begin{tikzcd} 0 \arrow[r] & C^n(X,Y) \arrow[r] \arrow[d, "\text{restriction}"'] & C^n(X) \arrow[r] \arrow[d, "\text{restriction}"'] & C^n(Y) \arrow[r] \arrow[d, "\text{restriction}"'] & 0 \\ 0 \arrow[r] & C^n(A+B, C+D) \arrow[r] & C^n(A+B) \arrow[r] & C^n(C+D) \arrow[r] & 0 \end{tikzcd}$$

where $C^n(A+B, C+D)$ is defined to be whatever makes the bottom row a s.e.s. Then the 5-Lemma says that the left hand map induces an isomorphism on cohomology. We also have

$$\begin{tikzcd} 0 \arrow[r] & C^*(A+B, C+D) \arrow[r] & C^*(A,C)\oplus C^*(B,D) \arrow[r] & C^*(A\cap B, C\cap D) \arrow[r] & 0 \end{tikzcd}$$

is a s.e.s of complexes. Then the associated l.e.s. in cohomology <u>is</u> the relative MV sequence we are after [**Exercise** to check].

□

Remark:(Footnote (viii)).

We claimed that if (Y,Z) is a pair such that $H^*(Y,Z)$ is free and finitely generated, then $H^*(X)\otimes H^*(Y,Z)\cong H^*(X\times Y, X\times Z)$ are isomorphic.

The Künneth theorem tells us that if $H^*(Y)$ is also free and finitely generated, then $H^*(X)\otimes H^*(Y)\cong H^*(X\times Y)$ are isomorphic. Thus we are essentially trying to prove a relative version of the Künneth theorem.

To prove this, we said to just apply the 5—Lemma. However there is an intermediate step that is needed. Observe that there is a commutative diagram:

$$\begin{tikzcd} H^*(X)\otimes H^*(Y,Z) \arrow[r, "\times"] & H^*(X\times Y, X\times Z) \\ H^*(X)\otimes H^*(Y/Z, \{\text{point}\}) \arrow[r, "\times"] \arrow[u] & H^*(X\times (Y/Z), X\times\{\text{point}\}) \arrow[u] \end{tikzcd}$$

where the × maps are the cross product, and the vertical maps come from the projection $Y\to Y/Z$.

Recall also that for good pairs, $H^*(P,Q) \cong \tilde{H}^*(P/Q)$. Now the spaces

$$\frac{X\times Y}{X\times Z} \quad \text{and} \quad \frac{X\times (Y/Z)}{X\times \{\text{point}\}}$$

are homeomorphic, and so it suffices to prove the relative cross product is an isomorphism in the special case that $Z = \{\text{point}\} \subset Y$. But in this case,

$$H^*(Y,\{\text{point}\}) \to H^*(Y) \to H^*(\{\text{point}\})$$

is a split exact sequence (via the maps $\{\text{point}\} \hookrightarrow Y \hookrightarrow \{\text{point}\}$). Then we can conclude here by the 5—Lemma.

☐

Note: Our proofs apply when our spaces have homotopy types of good pairs, e.g.

$$(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \simeq \left(\overline{D}^n(1)\, ,\, \overline{D}^n(1)\backslash\overline{D}^n(1/2)\right)$$

is a good pair model (where $\overline{D}^n(r)$ is the closed unit ball in $\mathbb{R}^n$ of radius r centred at the origin).

7.2. **Gysin Sequence.**

Let $E \xrightarrow{\pi} X$ be a vector bundle.

Vector bundles are a special case of the more general object known as a **fibre bundle**: $S \xrightarrow{p} X$ is a fibre bundle if p is a continuous surjection such that $p^{-1}(x) \cong F$ for all $x \in X$, where F is any topological space (not necessarily a vector space), called the **fibre**, and such that we have the usual locally trivial condition, i.e. for all $x \in X$, $\exists U \subset X$ open with $x \in U$ and a homeomorphism φ such that:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong", "\varphi"'] \arrow[dr, "p"] & & U\times E \arrow[dl, "\text{projection}"] \\ & U & \end{tikzcd}$$

commutes.

Definition 7.9. *We define the* ***sphere bundle*** *of E, denoted $S(E)$, or SE, to be the fibre bundle over X with fibres $F \cong S^{n-1}$ where $\operatorname{rank}_{\mathbb{R}}(E) = n$*

Note: $E^{\#} \simeq S(E)$ via inclusion and radial projection.

From everything we have seen and this homotopy equivalence we see that we have a l.e.s:

$$\begin{tikzcd} {} \arrow[r] & H^{i+n}(E,E^{\#}) \arrow[r] & H^{i+n}(E) \arrow[r] & H^{i+n}(E^{\#}) \arrow[r] & H^{i+n+1}(E,E^{\#}) \arrow[r] & \cdots \\ {} \arrow[r] & H^{n}(X) \arrow[r, "\varphi"'] \arrow[u, "\text{Thom isomorphism}", "\cong"'] & H^{i+n}(X) \arrow[r, "\pi^*"'] \arrow[u, "\text{bundle map}", "\cong"'] & H^{i+n}(S(E)) \arrow[r, "\pi_!"'] \arrow[u, "\text{homotopy inv.}", "\cong"'] & H^{i+1}(X) \arrow[r] \arrow[u, "\text{Thom isomorphism}", "\cong"'] & \cdots \end{tikzcd}$$

Here, $\varphi : H^i(X) \to H^{i+n}(X)$ is the cup product with the Euler class $e_E \in H^n(X)$, whilst π is the sphere bundle projection map $\pi : S(E) \to X$. The map $\pi_!$ we shall discuss later, but is pronounced "π lower shriek" and is what the bottom row needs to be exact and for the diagram to commute.

In more detail, the map φ takes (from commutativity of the diagram):

$$\alpha \underset{T}{\longmapsto} (\pi^*\alpha)\cdot u_E \longrightarrow (\pi^*\alpha)\cdot u_E|_E \xrightarrow{(\text{inclusion}_X)^*} \underbrace{\left(\text{inclusion}_X^*(\pi^*\alpha)\right)}_{=\alpha}\cdot \underbrace{u_E|_X}_{=e_E}$$

where T is the Thom isomorphism.

Definition 7.10. *The exact sequence*

$$\cdots \longrightarrow H^i(X) \xrightarrow{\cdot e_E} H^{i+n}(X) \longrightarrow H^{i+n}(S(E)) \longrightarrow H^{i+1}(X) \longrightarrow \cdots$$

is called the ***Gysin sequence*** *(pronounced "jee-sin").*

Note: Here n is the rank of the bundle E.

Example 7.7. *Let $\mathscr{L} \to \mathbb{C}P^n$ be the tautological complex line bundle. This is canonically oriented (see Example Sheet 3) and moreover it has sphere bundle*

$$S(\mathscr{L}) = S^{2n+1} \subset \mathbb{C}^n.$$

So the Gysin sequence becomes:

$$H^{i+1}(S^{2n+1}) \longrightarrow H^i(\mathbb{C}P^n) \xrightarrow{\cdot e_{\mathscr{L}}} H^{i+2}(\mathbb{C}P^n) \longrightarrow H^{i+2}(S^{2n+1}) \longrightarrow \cdots$$

So if $i \leq 2n-2$, this gives:

$$0 \longrightarrow H^i(\mathbb{C}P^n) \xrightarrow{\cdot e_{\mathscr{L}}} H^{i+2}(\mathbb{C}P^n) \longrightarrow 0$$

is exact, and so we see that multiplication by $e_{\mathscr{L}}$ is an isomorphism. Hence by induction this shows that the group $H^{2j}(\mathbb{C}P^n)$ is generated by $e^j_{\mathscr{L}}$, for all $1 \leq j \leq n$. So we get (since we know what the cohomology of $\mathbb{C}P^n$ is already, except now we have found canonical generators for the non-zero groups):

$$H^*(\mathbb{C}P^n) = \frac{\mathbb{Z}[e_{\mathscr{L}}]}{(e^{n+1}_{\mathscr{L}})}.$$

Compare this with question 6 on Example Sheet 2.

Remark: All the groups in the Gysin sequence are naturally modules for $H^*(X)$, since if $\pi : SE \to X$, then π^* makes $H^*(SE)$ into an $H^*(X)$-module.

Exercise: Show that the Gysin sequence is an exact sequence of $H^*(X)$-modules.

Example 7.8. *Let $V_k(\mathbb{C}^n)$ be the* ***Stiefel manifold*** *of ordered k—tuples of orthonormal vectors in $\mathbb{C}^n$. This is clearly a subspace of $\underbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}_{k \text{ times}}$, and $\exists$ a tautoloigcal complex vector bundle $E \to V_k(\mathbb{C}^n)$, where*

$$E_{\langle e_1,\ldots,e_k\rangle} := span\langle e_1, \ldots, e_k\rangle \subset \mathbb{C}^n.$$

Again, $E \subset V_k(\mathbb{C}^n) \times \mathbb{C}^n$ is a subbundle of the trivial bundle, and is locally trivial (this is similar to the proof of the tautological bundle for the Grassmannians $E_{\text{taut}} \to \text{Gr}_k(\mathbb{C}^n)$ begin locally trivial).

Proposition 7.1. *We have*

$$H^*(V_k(\mathbb{C}^n)) \cong \bigwedge_{\mathbb{Z}}(a_{2n-2k+1}, a_{2n-2k+3}, \ldots, a_{2n-3}, a_{2n-1})$$

is the exterior algebra generated by these $a_i \in H^i(V_k(\mathbb{C}^n))$ (i.e. the ring is the exterior algebra on these generators, which is free except for the relations imposed by graded commutativity).

Proof. We will prove this by induction on k. When $k = 1$, we have $V_1(\mathbb{C}^n) = S^{2n-1}$ and we know that $H^*(S^{2n-1}) = \wedge_{\mathbb{Z}}(a_{2n-1})$ and so the result is true here.

So suppose that the result is true for $V_k(\mathbb{C}^n)$. Note that there is a "forgetful map" $V_{k+1}(\mathbb{C}^n) \to V_k(\mathbb{C}^n)$ defined by

$$\langle e_1, \ldots, e_{k+1}\rangle \longmapsto \langle e_1, \ldots, e_k\rangle.$$

So let $F \to V_k(\mathbb{C}^n)$ be the bundle with fibre at $\langle e_1, \ldots, e_k\rangle$ being $F_{\langle e_1,\ldots,e_k\rangle} := \langle e_1, \ldots, e_k\rangle^{\perp}$. So then $V_{k+1}(\mathbb{C}^n) \equiv S(F)$, the unit sphere bundle F which has an Euler class $e_F \in H^{2n-2k}(V_k(\mathbb{C}^n)) = \{0\}$ (this is zero by the inductive hypothesis of the cohomology only being 0 in odd entries).

So the Gysin sequence becomes:

$$0 \xrightarrow{\cdot e_F} H^i(V_k) \longrightarrow H^i(V_{k+1}(\mathbb{C}^n)) \longrightarrow H^{i-2n+2k+1}(V_k(\mathbb{C}^n)) \xrightarrow{\cdot e_F} 0$$

since $e_F = 0$, where for the middle group we have used that $S(F) = V_{k+1}(\mathbb{C}^n)$ (this has fibres $S^{2n-2k+1}$).

Now <u>choose</u> $a_{2n-2k-1} \in H^{2n-2k-1}(V_{k+1}(\mathbb{C}^n))$ which maps to $1 \in H^0(V_k(\mathbb{C}^n))$ (which exists since setting $i = 2n - 2k - 1$ in the above we get $H^{2n-2k-1}(V_{k+1}(\mathbb{C}^n)) \to H^0(V_k(\mathbb{C}^n)$, and this is surjective by exactness).

Then the map $H^*(V_k(\mathbb{C}^n)) \oplus H^*(V_k(\mathbb{C}^n)) \to H^*(V_{k+1}(\mathbb{C}^n))$ sending

$$(\alpha, \beta) \longmapsto \alpha + a_{2n-2k-1} \cdot \beta$$

is an isomorphism (both additively and as a map of $H^*(V_k(\mathbb{C}^n))$—modules). The result then follows by induction.

□

Remark: The map $\pi_! : H^{*+n}(SE) \to H^{*+1}(X)$ is often called **integration over the fibre**. When working with de Rham cohomology, $\pi_!$ is exactly this: it is integration of differential forms over fibres.

Corollary 7.2. *We have that the cohomology of $U(n)$ is:*

$$H^*(U(n)) := \bigwedge_{\mathbb{Z}}(a_1, a_3, \ldots, a_{2n-1}).$$

In particular, if $b_i(U(n)) = rank\big(H^i(U(n))\big)$ *is the rank, then:*

$$\sum_{i\geq 0} b_i(U(n))t^i = \prod_{i=1}^{n}(1+t^{2i-1}).$$

Remark: The rank of the cohomology groups are usually called the **Betti numbers** of the space.

Proof. Simply note that $V_n(\mathbb{C}^n) = U(n)$, so the first claim follows from Proposition 7.1 applied when $k = n$.

For the second part, this is just an identity via partitions.

□

We have now studied $H^*(E, E^{\#})$ inductively over a trivialising cover. We would like, analogously, to study the cohomology $H^*(M)$ of a (closed) manifold M inductively over a cover (e.g. the cover coming from an atlas of charts). We do this via **cohomology with compact support**.

7.3. **Cohomology with Compact Support.**

Definition 7.11. *Let A be a poset such that:*

$$(\dagger) \qquad \text{For all } a, b \in A,\ \exists c \text{ s.t. } c \geq a \text{ and } c \geq b.$$

Then a ***direct system*** *of groups indexed by A is a collection* $(G_a)_{a\in A}$ *of (abelian) groups and homomorphisms* $\rho_{ab} : G_a \to G_b$ *whenever* $a \leq b$ *(i.e. the* ρ *maps "go up the system") such that:*

- $\rho_{aa} = \mathrm{id}|_{G_a}$,
- $\rho_{ab} \circ \rho_{bc} = \rho_{ac}$

for all $a \leq b \leq c$.

The ***direct limit*** *of a direct system is then defined by:*

$$\varinjlim_{A} G_a := \frac{\bigoplus_{a\in A} G_a}{\langle \alpha - \rho_{ab}(\alpha)\rangle}$$

i.e. $\alpha \sim \rho_{ab}(\alpha)$ *for all* $\alpha \in G_a$ *and all* $a \leq b$. *So we identify* α *with all of its images under* ρ_{ab} *maps.*

Via (†) we see that this direct limit inherits a group structure in the following way:

If $\alpha \in G_a$ and $\beta \in G_b$, then $\exists c$ s.t. $\alpha \sim \alpha' \in G_c$ and $\beta \sim \beta' \in G_c$ and then set: $[\alpha+\beta] := [\alpha' + \beta']$.

This is then well-defined by the assumptions of a direct system.

Now if X is a space and K_1, K_2 are compact subsets of X with $K_1 \subset K_2$, then we have $X\backslash K_2 \subset X\backslash K_1$ and so $\exists$ a inclusion of pairs $(X, X\backslash K_2) \hookrightarrow (X, X\backslash K_1)$, which gives a map:

$$H^*(X, X\backslash K_1) \to H^*(X, X\backslash K_2)$$

[i.e. taking complements flips the inclusion, but then as cohomology is contravariant this flips the arrow again, and thus we get the right direction].

Hence we have a direct system $H_K := H^*(X, X\backslash K)$ for $K \in \mathscr{K}$, where $\mathscr{K}$ is the poset:

$$\mathscr{K} := \{K \subset X : K \text{ is a compact subset of } X\}$$

and elements of $\mathscr{K}$ are ordered by inclusion. The homomorphisms are then induced by the inclusion maps as above.

Then we can define:

Definition 7.12. *We define the* ***cohomology of X with compact support*** *by the direct limit:*

$$H^*_{\text{ct}}(X) := \varinjlim_{\mathscr{K}} H^*(X, X\backslash K).$$

We will study this for awhile and see how it helps us compute the cohomology of manifolds.

Remark: In general if $(G_a)_{a\in A}$ is a direct system and $A' \subset A$ is such that for all $a \in A$, $\exists a' \in A'$ such that $a \leq a'$, then we have:

$$(\star) \qquad \varinjlim_{A} G_a = \varinjlim_{A'} G_{a'}.$$

Example 7.9. *We have that:*

$$H^*_{\text{ct}}(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \textit{if } * = n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $K \subset \mathbb{R}^n$ is compact, then $\exists n \in \mathbb{N}$ such that $K \subset \overline{B}_N(0)$. So by $(\star)$ abpve we have

$$\varinjlim_{\mathscr{K}} H^*(\mathbb{R}^n, \mathbb{R}^n\backslash K) = \varinjlim_{N\in\mathbb{N}} \underbrace{H^*(\mathbb{R}^n, \mathbb{R}^n\backslash \overline{B}_N(0))}_{=H^*(S^{n-1}) \quad \forall N}$$

with the equality on the RHS in a compatible way for each N. Hence we see that (as the RHS is a constant):

$$\varinjlim_{\mathscr{K}} H^*(\mathbb{R}^n, \mathbb{R}^n\backslash K) = \{\cdots \to \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{\text{id}} \cdots\}$$

which is $\mathbb{Z}$ in degree n and zero otherwise. □

Example 7.10. *If X is compact, then every compact set in X is contained in X, and so applying $(\star)$ to $A' = \{X\}$ we see that*

$$H^*_{ct}(X) = \varinjlim_{\mathscr{K}} H^*(X, X\backslash K) = H^*(X, X\backslash X) = H^*(X)$$

i.e. for compact sets X we have $H^*_{ct}(X) = H^*(X)$. In particular if we take $X = \{point\}$ or $X = \overline{B}_1(0)$ then we see:

$$H^*_{ct}(X) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Combining Example 7.10 with Example 7.9 we see that H^*_{ct} is <u>not</u> homotopy invariant.

Indeed, a general map $f : X \to Y$ does <u>not</u> induce a map on H^*_{ct}. However there are two cases when it does:

(i) If $f : X \to Y$ is a **proper** map, i.e. f is a closed map and the pre-image of a compact set is compact under f, then f <u>does</u> induce a map

$$H^*_{\text{ct}}(Y) \xrightarrow{f^*} H^*_{\text{ct}}(X).$$

(ii) If $i : U \hookrightarrow X$ is the inclusion of an open subset $U \subset X$, then there is an "extension by zero" pushforward map $i_* : H^*_{\text{ct}}(U) \to H^*_{\text{ct}}(X)$(ix). Here, if $K \subset U$ is compact, then $H^*(U, U\backslash K) \cong H^n(X, X\backslash K)$ by excision. Since X has more compact sets than U, then $\exists$ a map

$$\varinjlim_{K \subset U \text{ compact}} H^*(U, U\backslash K) \longrightarrow \varinjlim_{\tilde{K} \subset X \text{ compact}} H^*(X, X\backslash \tilde{K})$$

which gives the map on cohomology with compact support we were after.

Remark: We can think of H^*_{ct} coming from a "chain complex" via:

$$C^*_{\text{ct}}(X) := \bigcup_{K \subset X \text{ compact}} C^*(X, X\backslash K)$$

however we won't use this.

Example 7.11. *If $i : U \hookrightarrow \mathbb{R}^n$ is the inclusion of an open disc, then*

$$i_* : H^*_{ct}(U) \longrightarrow H^*_{ct}(\mathbb{R}^n)$$

is an isomorphism.

Now because we want to use cohomology with compact supports to study manifolds, and currently this works locally we need some kind of gluing lemma/Mayer-Vietoris. This is the following:

Proposition 7.2 (Mayer-Vietoris for Cohomology with Compact Support)**.** *Let X be a locally compact space*(x)*. Let $X = U \cup V$ be a union of open subsets. Then $\exists$ a Mayer-Vietoris sequence:*

$$\cdots \longrightarrow H^{i-1}_{ct}(X) \longrightarrow H^i_{ct}(U \cap V) \longrightarrow H^i_{ct}(U) \oplus H^i_{ct}(V) \longrightarrow H^i_{ct}(X) \longrightarrow H^{i+1}_{ct}(U \cap V) \longrightarrow$$

(ix) i.e. if a function has compact support in U, then it can be extended to a function with compact support in X by setting it to be zero on $X\backslash U$.

Remark: Note the direction of the arrows in this Mayer-Vietoris sequence!

Proof. Recall that if $(X,Y)=(A\cup B, C\cup D)$ isa. union of pairs, we had a relative Mayer-Vietoris sequence:

$$\longrightarrow H^n(X,Y)\longrightarrow H^n(A,C)\oplus H^n(B,D)\longrightarrow H^n(A\cap B, C\cap D)\longrightarrow H^{n+1}(X,Y)\longrightarrow\cdots$$

So suppose $K\subset U$ and $L\subset V$ are compact. Let $A=B=X$, $C=X\backslash K$ and $D=X\backslash L$ in the above. Then set $Y:=C\cup D=X\backslash(K\cap L)$, and note that $C\cap D=X\backslash(K\cup L)$. Then the above MV sequence gives:

$$H^i(X,X\backslash(K\cap L))\longrightarrow H^i(X,X\backslash K)\oplus H^i(X,X\backslash L)\longrightarrow H^i(X,X\backslash(K\cup L))\longrightarrow\cdots$$

We can then excise $X\backslash(U\cap V)$, $X\backslash U$ and $X\backslash V$ from the relevant groups in the above to obtain:

$$H^i(U\cap V,(U\cap V)\backslash(K\cap L))\longrightarrow H^i(U,U\backslash K)\oplus H^i(V,V\backslash L)\longrightarrow H^i(X,X\backslash(K\cup L))\longrightarrow\cdots$$

A compact set in $U\cap V$ is always of the form $K\cap L$ for $K\subset U$ compact and $L\subset V$ compact. Take K,L to be the given compact set.

Now note that every compact set in X is contained in some $K\cup L$ for $K\subset U$ compact and $L\subset V$ compact (this is true since X is locally compact).

Now if $C\subset X$ is compact, then it admits a finite cover by compact sets each contained in U or V, and whose interiors cover C.

Then since the direct limit of exact sequences is exact [**Exercise** - see Example Sheet 4], by passing to the direct limits $\varinjlim_{K\subset U\text{ compact}}$ and $\varinjlim_{L\subset V\text{ compact}}$ in the above, we get exactly the required sequence.

□

Definition 7.13. *Let M be a manifold. Then we say M has* ***finite type*** *if we can write*

$$M=\bigcup_{i=1}^{N}U_i$$

such that each U_i and each iterated intersection $U_{i_1}\cap\cdots\cap U_{i_j}$ (for $i_1<\cdots<i_j$, $j=1,\dots,N$) is either an open ball (i.e. homeomorphic to $\mathbb{R}^n$) or is empty.

We then say that $(U_i)_{i=1}^{N}$ is a ***good cover*** *of M.*

Fact: If M is a closed smooth manifold, or the interior of a compact smooth manifold with boundary, then M has finite type.

[This can be proved via, given a Riemannian metric on M, sufficiently small metric balls in M are geodesically convex (so their interiors are convex and so homeomorphic to a point, and so to $\mathbb{R}^n$. Also the intersection of convex sets is still convex, so we also get this for the iterated intersections) - See Differential Geometry Part III.]

(x) As we will be talking about manifolds, this will always be assumed, even if we don't say it.

Lemma 7.3. *Let M be a manifold of finite type. Then:*

(i) $H^i_{\text{ct}}(M) = 0$ *for all* $i > n = \dim_{\mathbb{R}}(M)$.

(ii) $H^i_{\text{ct}}(M)$ *is a finitely generated group for all* i.

(iii) If M *is connected, then* $H^n_{\text{ct}}(M)$ *is a cyclic group.*

Indeed, if $i : U \hookrightarrow M$ *is the inclusion of an open disc, then* $i_* : H^n_{\text{ct}}(U) \twoheadrightarrow H^n_{\text{ct}}(M)$ *is surjective (note that we know* $H^n_{\text{ct}}(U) \cong \mathbb{Z}$*).*

Proof. We shall prove this by inducting on N, the number of sets in a good cover of M.

If $N = 1$, then $M \cong \mathbb{R}^n$ and then the results are clear by Example 7.9.

Now suppose $M > 1$. Then we can write $M = U_0 \cup U_1$, where $U_0 \cong \mathbb{R}^n$ and $U_1, U_0 \cap U_1$ have smaller "type". Then for (i), the Mayer-Vietoris sequence for cohomology of compact support becomes:

$$\cdots \longrightarrow H^i_{\text{ct}}(U_0 \cap U_1) \longrightarrow H^i_{\text{ct}}(U_0) \oplus H^i_{\text{ct}}(U_1) \longrightarrow H^i_{\text{ct}}(M) \longrightarrow H^{i+1}_{\text{ct}}(U_0 \cap U_1) \longrightarrow \cdots$$

and so we get, if $i > n$,

$$0 \longrightarrow H^i_{\text{ct}}(M) \longrightarrow 0$$

which proves (i).

Similarly, we know that if H and G/H are finitely generated abelian groups, then G is also finitely generated. So (ii) then follows.

Now take $i = n$ in the above MV sequence. This gives

$$H^n_{\text{ct}}(U_0 \cap U_1) \longrightarrow H^n_{\text{ct}}(U_0) \oplus H^n_{\text{ct}}(U_1) \longrightarrow H^n_{\text{ct}}(M) \longrightarrow \underbrace{H^{n+1}_{\text{ct}}(U_0 \cap U_1)}_{=0} \longrightarrow \cdots$$

Now since M is connected we must have $U_0 \cap U_1 \neq \emptyset$, since otherwise M would be the union of two disjoint open sets. So we can find a disc $D \subset U_0 \cap U_1 \subset U_0$, which induces via the inclusion a map $H^n_{\text{ct}}(D) \to H^n_{\text{ct}}(U_0)$ in the usual way ("extension by zero").

Hence $H^n_{\text{ct}}(U_0 \cap U_1) \to H^n_{\text{ct}}(U_0)$ is certainly onto, and so exactness in the above sequence gives that $H^n_{\text{ct}}(U_1) \to H^n_{\text{ct}}(M)$ is onto. So hence by the first isomorphism theorem we see that $H^n_{\text{ct}}(M)$ is a quotient of a cyclic group, and so is itself cycle.

If we then consider $D \hookrightarrow U_1 \hookrightarrow M$, we then inductively (i.e. apply induction hypothesis on $D \hookrightarrow U_1$) see that $H^n_{\text{ct}}(D) \to H^n_{\text{ct}}(M)$ is onto (i.e. get $H^n_{\text{ct}}(D) \to H^n_{\text{ct}}(U_1)$ is onto as well, and so can compose).

□

Remark: If M is compact, so that $H^*_{\text{ct}}(M) \equiv H^*(M)$, we see by the above that $\underline{H^*(M) \text{ is a finitely generated abelian group}}$, and is non-zero only if $* \in \{0, \ldots, n\}$.

Definition 7.14. *Let M be a (topological) manifold. Then we say M is* ***oriented*** *if for all discs $\mathbb{R}^n \cong U \subset M$, we can $\underline{\text{fix}}$ a generator $\omega_U \in H^n_{ct}(U)$ such that whenever $\underset{\cong\mathbb{R}^n}{U} \subset \underset{\cong\mathbb{R}^n}{V} \subset M$ we have $i_*\omega_U = \omega_V$ (i.e. compatible under restrictions).*

Remarks:

(i) We say a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is **orientation-preserving** if $f_* : H^n_{\text{ct}}(\mathbb{R}^n) \to H^n_{\text{ct}}(\mathbb{R}^n)$ is multiplication by $+1$ (and it is **orientation-reversing** if it is multiplication by -1.) Then we can see that:

$$M \text{ is orientable} \iff \text{It admits an orientation-preserving atlas.}$$

(ii) $H^n_{\text{ct}}(\mathbb{R}^n)$ is naturally isomorphic to $H^n(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$ (look back at our calculation of $H^n(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$). So we could take orientation generators $\omega_U \in H^n(U, U\backslash\{x\})$ for $x \in U$.

Since excision identifies $H^n(U, U\backslash\{x\}) \cong H^n(M, M\backslash\{x\})$, we can also say that M is orientated if we have generators $\mu_x \in H^n(M, M\backslash\{x\})$ for all $x \in M$, which vary locally trivially.

(iii) We can also take $\mu_x \in H_n(M, M\backslash\{x\}) \cong H_n(U, U\backslash\{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$.

(iv) If M is a smooth manifold, the exponential map for a Riemannian metric identifies small discs in T_xM with small discs $x \in U \subset M$, so the above definition of M being orientable agrees with the definition:

"M is orientable if TM is orientable as a vector bundle"

for smooth manifold M (since then the tangent bundle TM is defined).

Theorem 7.2. *Let M be a connected manifold of finite type. Then:*

(i) If M is oriented, $\exists!$ isomorphism $H^n_{ct}(M) \xrightarrow{\cong} \mathbb{Z}$ such that if $U \subset M$ is an open disc, the resulting map $H^n_{ct}(U) \to H^n_{ct}(M) \cong \mathbb{Z}$ sends $\omega_U \mapsto 1$.

(ii) If M is not orientable, then $H^n_{ct}(M) \cong \mathbb{Z}_2$.

Remark: In de Rham cohomology, the map $H^n_{\text{ct,dR}}(M) \xrightarrow{\cong} \mathbb{R}$ is given by $\int_M$, i.e. integration of n-forms on M.

Proof. Since M has finite type, write $M = \bigcup_{i=1}^N U_i$ for a finite good cover $\{U_i\}_i$. Then let $W_i = U_1 \cup \cdots \cup U_i$. We shall work by induction on N. The base case when $M = U_1 \cong \mathbb{R}^n$ is clear.

Inductively, first suppose W_i is orientated. Then write

$$W_i \cap U_{i+1} = V_1 \amalg \cdots \amalg V_p$$

as a disjoint union, where the $(V_i)_i$ are the path components and are of lower type and sit inside the (orientable) disc U_{i+1} (the V_j need not be discs themselves).

Then Mayer-Vietoris gives:

$$H^n_{\mathrm{ct}}(V_1) \oplus \cdots \oplus H^n_{\mathrm{ct}}(V_p) \xrightarrow{\varphi} H^n_{\mathrm{ct}}(W_i) \oplus H^n_{\mathrm{ct}}(U_{i+1}) \xrightarrow{\alpha} H^n_{\mathrm{ct}}(W_{i+1}) \longrightarrow 0$$

and so since $H^n_{\mathrm{ct}}(W_i) \cong \mathbb{Z}$ by induction, this gives

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} H^n_{\mathrm{ct}}(W_{i+1}) \longrightarrow 0.$$

Let $\omega_i \in H^n_{\mathrm{ct}}(V_i)$ be a generator for each i, chosen such that $\varphi(\omega_i) = (1, \varepsilon_i)$, with $\varepsilon_i = +1$ or -1. (Note that $1 \in H^n_{\mathrm{ct}}(W_i)$ is the orientation generator.)

Then there are two possibilities:

(a) All the ε_i agree. Then we can define an orientation [**Exercise** to check] on U_{i+1} such that $\varphi(\omega_i) = (1,1)$ for all i (where the second 1 is the choice of orientation on U_{i+1}). Then W_{i+1} inherits a coherent orientation and exactness gives $H^n_{\mathrm{ct}}(W_{i+1}) \cong \mathbb{Z}$.

(b) Not all the ε_i agree. Then, by the first isomorphism theorem

$$H^n_{\mathrm{ct}}(W_{i+1}) \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (1,1), (1,-1) \rangle} \cong \mathbb{Z}_2.$$

Now if M itself is orientated, then $W_j \subset M$ is orientated for all j, and so we are done in this case by (a) (as we $H^n_{\mathrm{ct}}(W_N) \cong \mathbb{Z}$ and $W_N = M$).

So it remains to show that if M is not orientable, and, say, $i+1$ is the first index such that $H^n_{\mathrm{ct}}(W_{i+1}) \cong \mathbb{Z}_2$, then $H^n_{\mathrm{ct}}(W_j) \cong \mathbb{Z}$ for all $j \geq i+1$.

Using the same analysis as the above, if W_j is not orientable, and W_{j+1} is the next subspace with $W_j \cap U_{j+1} = \tilde{V}_1 \amalg \cdots \amalg \tilde{V}_q$, the analogous Mayer-Vietoris sequence is:

$$H^n_{\mathrm{ct}}(\tilde{V}_1) \oplus \cdots \oplus H^n_{\mathrm{ct}}(\tilde{V}_q) \xrightarrow{\varphi} H^n_{\mathrm{ct}}(W_j) \oplus H^n_{\mathrm{ct}}(U_{j+1}) \longrightarrow H^n_{\mathrm{ct}}(W_{j+1}) \longrightarrow 0$$

i.e. we have by induction $H^n_{\mathrm{ct}}(W_j) \cong \mathbb{Z}_2$ due to non-orientablity, and so

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_2 \oplus \mathbb{Z} \longrightarrow H^n_{\mathrm{ct}}(W_{j+1}) \longrightarrow 0.$$

Now pick generators $\tilde{\omega}_i \in H^n_{\mathrm{ct}}(\tilde{V}_i)$ such that $\varphi(\tilde{\omega}_i) = (\tilde{\varepsilon}_i, 1)$ with respect to a choice of orientation for U_{j+1}. Then Lemma 7.3(iii) says that $\tilde{V}_i \hookrightarrow W_j$ is surjective on H^n_{ct} for each j, and $\tilde{\varepsilon}_j = 1 \in \mathbb{Z}_2 = \{0,1\}$ for all i.

Then finally, exactness implies that $H^n_{\mathrm{ct}}(W_{j+1}) \cong \mathbb{Z}_2$, and so by induction we get this is true for all $j \geq i+1$, and so we are done by taking $i = N$.

□

8. POINCARÉ DUALITY

8.1. Cohomology Classes of Submanifolds.

Our next goal is to give a geometric interpretation of the cup product on a smooth manifold. Recall that if M is a smooth manifold it has a tangent bundle TM, and if $Y \subset M$ is a smooth submanifold, then for all $y \in Y$ we have $T_yY \subset T_yM$, and the quotient:

$$\frac{T_yM}{T_yY} =: (\nu_{Y/M})_y$$

is the fibre of the **normal bundle** $\nu_{Y/M}$ of Y in M. This has (real) rank $\operatorname{codim}_M(Y)$, the codimension of Y in M.

[If g is a metric on M or TM, then in fact we have $\nu_{Y/M} \cong T^{\perp}Y \equiv (TY)^{\perp} \subset TM$, where $\perp$ here denotes the orthogonal complement with respect to the metric/inner product on the fibres.]

Definition 8.1. *We say that a submanifold $Y \subset M$ is **co-oriented** if $\nu_{Y/M}$ is oriented.*

Remark: On Example Sheet 3, you will prove that if Y and M are both oriented, then Y is naturally co-oriented.

An important result from differential topology which we will make use of (but not prove) is the **tubular neighbourhood theorem**:

Theorem 8.1 (The Tubular Neighbourhood Theorem)**.** *Let M be a smooth manifold. Then:*

(i) If $Y \subset M$ is a closed smooth submanifold, then there is an open neighbourhood $U_Y \subset M$ of Y in M and a diffeomorphism $\varphi : U_Y \to \nu_{Y/M}$ such that the following diagram commutes:

$$\begin{tikzcd} U_Y \arrow[r, "\varphi", "\cong"'] & \nu_{Y/M} \\ Y \arrow[u, hook] \arrow[r, "\mathrm{id}", "\cong"'] & 0_Y \arrow[u, hook] \end{tikzcd}$$

where the vertical arrows are inclusion maps and 0_Y is the zero-section over Y.

Moreover, both U_Y and φ are unique up to isotopy.

(ii) If $Y, Z \subset M$ are closed smooth submanifold which intersect transversely$^{(\mathrm{xi})}$, then $Y \cap Z$ is a smooth submanifold, and

$$\operatorname{codim}_M(Y \cap Z) = \operatorname{codim}_M(Y) + \operatorname{codim}_M(Z)$$

and

$$\nu_{Y\cap Z/M} \cong \nu_{Y/M}\big|_{Y\cap Z} \oplus \nu_{Z/M}\big|_{Y\cap Z}$$

and $\exists$ a tubular neighbourhood $U_{Y\cap Z} = U_Y \cap U_Z$ such that φ is compatible with this splitting.

Proof. None given. ☐

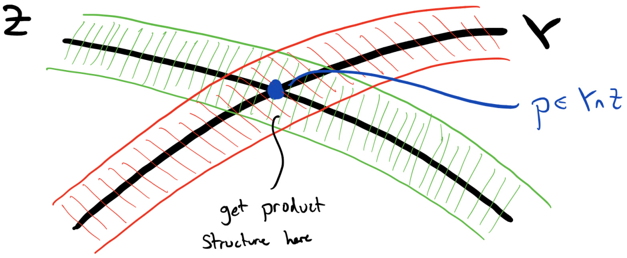


FIGURE 24. An illustration of the tubular neighbourhood theorem.

So let $Y^k \hookrightarrow M$ be a closed co-oriented smooth submanifold of dimension k of a smooth manifold M. Then by definition we know $\nu_{Y/M}$ is oriented, and so we have a Thom class

$$u_{\nu_{Y/M}} \in H^{n-k}(\nu_Y, \nu_Y^{\#}) = H^{n-k}(U_Y, U_Y \backslash Y) \cong H^{n-k}(M, M \backslash Y)$$

where the first equality is by the tubular neighbourhood theorem and the second by excision. Note that $H^{n-k}(M, M\backslash Y) \to H^{n-k}_{\text{ct}}(M)$, and thus we get an image of the Thom class $u_{\nu_{Y/M}}$ in $H^{n-k}_{\text{ct}}(M)$ under this map.

Definition 8.2. *Class the image of the Thom class $u_{\nu_{Y/M}}$ in $H^{n-k}_{\text{ct}}(M)$ the* ***cohomology class of Y*** *(or associated to Y). We denote it by $\varepsilon_Y \in H^{n-k}_{\text{ct}}(M) = H^{\text{codim}(Y)}_{\text{ct}}(M)$.*

Remark: As Y is closed, $i : Y \hookrightarrow M$ is a proper map, and so we can restrict $\varepsilon_Y|_Y := i^*\varepsilon_Y \in H^{n-k}_{\text{ct}}(Y) = H^{n-k}(Y)$, where the last equality here is because Y is closed and so compact. Then from the definitions we see that $\varepsilon_Y|_Y = e_{\nu_{Y/M}} \in H^{n-k}(Y)$ is in fact the Euler class (compare this with (Thom class)$|_{\text{zero-section}}$ = Euler class). [**Exercise** to check.]

Example 8.1. *If $Y = \{point\} \subset M$, then Y is co-oriented if M is oriented [**Exercise** to check]. Then $\varepsilon_{\{point\}} \in H^n_{ct}(M)$ is the orientation generator for the orientation on M.*

The main result for computing cup products on manifolds is:

Proposition 8.1. *Let $Y, Z \subset M$ be co-oriented, closed, smooth submanifolds of a closed manifold M which intersect transversely. Then:*

$$\varepsilon_{Y\cap Z} = \varepsilon_Y \cdot \varepsilon_Z \in H^{\text{codim}(Y)+\text{codim}(Z)}(M).$$

[So the cup product is "dual" to intersections.]

[xi] By intersect transversely we mean for all $x \in Y \cap Z$ we have $T_xM = T_xY + T_xZ$ (although this is not necessarily a direct sum, just a sum).

Remark: We know the cup product is skew-commutative, and so

$$\varepsilon_Y \cdot \varepsilon_Z = (-1)^{\text{codim}(Y)+\text{codim}(Z)} \varepsilon_Z \cdot \varepsilon_Y.$$

So is the above consistent? Well the definition of $\varepsilon_{Y\cap Z}$ depends on a co-orientation of $Y \cap Z$. A question on Example Sheet 3 shows that an $\underline{\text{ordering}}$ of Y, Z defines such a co-orientation, and we find:

$$\varepsilon_{Z\cap Y} = (-1)^{\text{codim}(Y)+\text{codim}(Z)} \varepsilon_{Y\cap Z}$$

and so the statement is consistent.

Proof. We have a relative cup product

$$H^i(X,A) \otimes H^j(Y,B) \to H^{i+j}(X \times Y, A \times Y \cup X \times B).$$

So suppose $E \to X$ and $F \to X$ are oriented vector bundles. Then the relative cup product defines a map

$$H^i(E,E^\#) \otimes H^j(F,F^\#) \longrightarrow H^{i+j}(E \oplus F, (E \oplus F)^\#) \quad \text{sending} \quad x \otimes y \longmapsto (\pi_E^* x) \cdot (\pi_F^* y).$$

If $X = \{\text{point}\}$ then the Künneth formula said

$$H^i(\mathbb{R}^i, \mathbb{R}^i \backslash \{0\}) \otimes H^j(\mathbb{R}^j, \mathbb{R}^j \backslash \{0\}) \to H^{i+j}(\mathbb{R}^{i+j}, \mathbb{R}^{i+j} \backslash \{0\})$$

is an isomorphism. This shows that $u_{E\oplus F} = \pi_E^* u_E \cdot \pi_F^* u_F$, since for any bundle $W \to X$ of rank d, u_W is (uniquely) characterised by being the unique class such that $u_W|_{(\mathbb{R}^d, \mathbb{R}^d \backslash \{0\})}$ was the generator ε_x of the fibre $(W_x, W_x^\#) = (\mathbb{R}^d, \mathbb{R}^d \backslash \{0\})$.

Now the transverse intersection of Y, Z says that $\nu_{Y\cap Z/M} = \nu_{Y/M} \oplus \nu_{Z/M}$ [**Exercise** to check], and then the result follows by the definition of $\varepsilon_{Y\cap Z}$.

☐

Corollary 8.1. *If Y, Z are oriented closed submanifolds of an oriented, closed, smooth manifold M, and if $Y \cap Z = \{point\}$ is a transverse intersection, then $\varepsilon_Y \cdot \varepsilon_Z = \pm 1 \in H^n(M) \cong \mathbb{Z}$. In particular, ε_Y and ε_Z are non-zero.*

Proof. Just apply Proposition 8.1 in the special case of $Y \cap Z = \{\text{point}\}$. ☐

Using this, we can just read off answers to many cup product-type questions we have done previously.

Example 8.2 (Calculating $H^*(\Sigma_2)$). *Consider the picture in Figure 25. The picutre shows us that, by Corollary 8.1, $\varepsilon_{a_i} \cdot \varepsilon_{b_j} = \delta_{ij}$, and hence $\{\varepsilon_{a_1}, \varepsilon_{b_1}, \varepsilon_{a_2}, \varepsilon_{b_2}\}$ are non-zero and linearly independent in $H^1(\Sigma_2)$. Hence the determine $H^*(\Sigma_2)$ as a ring (using skew-commutativity of the cup product).*

A similar idea can be used for Σ_g.

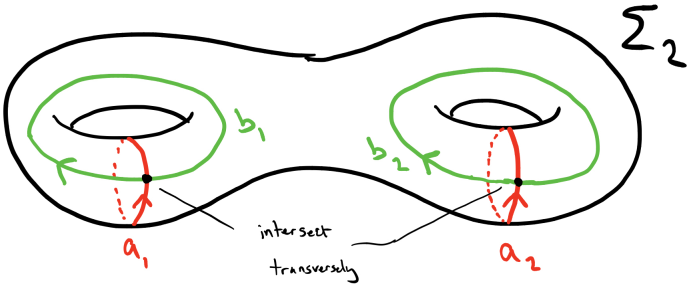


FIGURE 25. Illustration of Example 8.2.

Example 8.3 ($\mathbb{C}P^1 \times \mathbb{C}P^1$ versus $\mathbb{C}P^2 \# \mathbb{C}P^2$). *Recall that both these structures have additively isomorphic cohomology, and in both cases* $H^2 \cong \mathbb{Z} \oplus \mathbb{Z}$.

For $\mathbb{C}P^1 \times \mathbb{C}P^1$*, using the diagram in Figure 26 (on the LHS) we see that* $A \cap B = \{\text{point}\}$*, and so* ε_A *and* ε_B *form a basis of* $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1)$*. In this basis the cup product has the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For $\mathbb{C}P^2 \# \mathbb{C}P^2$*, using the diagram in Figure 26 (on the RHS) we have* $\varepsilon_\alpha \cdot \varepsilon_\alpha = 1$ *and* $\varepsilon_\beta \cdot \varepsilon_\beta = 1$ $(= \varepsilon_{\{\text{point}\}} \in H^4)$*, and so the corresponding symmetric matrix for the cup product in the* $\{\alpha, \beta\}$*—basis is* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Since these two matrices differ, these spaces cannot be homeomorphic. □

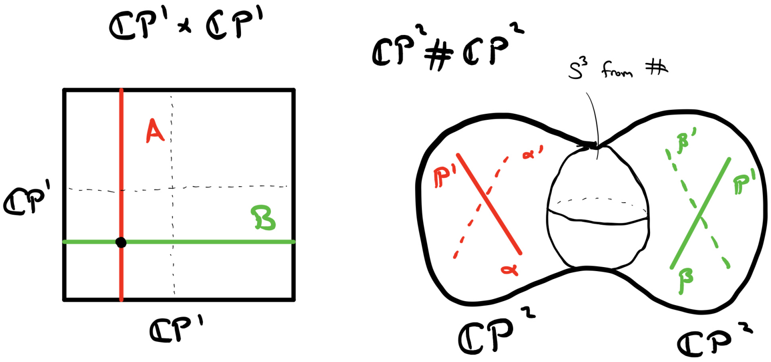


FIGURE 26. Illustration of Example 8.3.

Remark: Every manifold is $\mathbb{Z}_2$-oriented (in the obvious sense), and all bundles are oriented over $\mathbb{Z}_2$. So for mod 2 cup products, Corolllary 8.1 applies to all such M.

Remark: Thom (in the 1950's) showed that not every class $A \in H^j(M;\mathbb{Z})$ is realised by a codimension j submanifold. But if you work over the field $k = \mathbb{Q}$ or $\mathbb{Z}_2$, then $\{\varepsilon_Y : Y \text{ is a smooth submanifold of } M\}$ does span $H^*(M;k)$.

8.2. **Poincaré Duality.**

So far we could in principle have that the cup product on M is degenerate (as for ΣX, X any space). This however is ruled out by **Poincaré duality**, the main structural result in the cohomology of manifolds.

The relative cup product $H^i(X) \otimes H^j(X,A) \to H^{i+j}(X,A)$ yields a product $H^i(X) \otimes H^j(X,X\backslash K) \to H^{i+j}(X,X\backslash K)$. Then by naturality with respect to maps of spaces, we get a product

$$H^i(X) \otimes H^j_{\text{ct}}(X) \to H^{i+j}_{\text{ct}}(X).$$

This leads to:

Theorem 8.2 (Poincaré Duality - "Pairing" formulation.)**.** *Fix a field $\mathbb{F}$. Then if M is an $\mathbb{F}$-oriented manifold (of finite type), the cup product*

$$H^j(M,\mathbb{F}) \otimes H^{n-j}_{\text{ct}}(M,\mathbb{F}) \to H^n_{\text{ct}}(M,\mathbb{F}) \cong \mathbb{F}$$

(all with $\mathbb{F}$-coefficients) is non-degenerate.

In particular, $\left(H^j(M,\mathbb{F})\right)^* \cong H^{n-j}_{ct}(M,\mathbb{F})$.

Proof. Later. Note that (from the proof) this says that a cohomology class is completely determined by its cap products with things of 'complementary degree'.

□

Corollary 8.2. *Let M be a closed manifold of odd dimension. Then M has Euler characteristic $\chi(M) = 0$.*

Proof. Let $\mathbb{F} = \mathbb{Z}_2$. Then we know that M is always $\mathbb{F}$-oriented by a previous remark. We saw that we can compute $\chi(M)$ as the alternating sum (see Lemma 4.3):

$$\chi(M) = \sum_{i\geq 0}(-1)^i \text{rank}_{\mathbb{F}}(H^i(M,\mathbb{F}))$$

for any field $\mathbb{F}$. But Poincare duality says (since M is closed and so compact, meaning $H^*_{\text{ct}}(M) \cong H^*(M)$) $H^i(M,\mathbb{F}) \cong H^{n-i}(M,\mathbb{F})^*$, and so all the terms in this alternating sum cancel in pairs, since M is odd-dimensional. [Recall that $\dim(V^*) = \dim(V)$ for $\dim(V) < \infty$.]

□

Corollary 8.3. *Let M^n, N^n be closed $\mathbb{F}$-oriented manifolds and $f : M \to N$ a map. Let* $\deg_{\mathbb{F}}(f)$ *be the degree of the map $f^* : \underbrace{H^n(N;\mathbb{F})}_{\cong\mathbb{F}} \to \underbrace{H^n(M;\mathbb{F})}_{\cong\mathbb{F}}$, and suppose* $\deg_{\mathbb{F}}(f) \neq 0$. *Then f^* is injective.*

Proof. Let $\alpha \in H^i(N,\mathbb{F})$. Then if $\alpha \neq 0$, $\exists \beta \in H^{n-\alpha}(N;\mathbb{F})$ such that $\alpha \cdot \beta \in H^n(N,\mathbb{F})$, from the non-degeneracy established in Poincaré duality. Then since $\deg_{\mathbb{F}}(f) \neq 0$, we have

$$0 \neq f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta)$$

and so $f^*(\alpha) \neq 0$. So hence $\ker(f^*) = \{0\}$ and so f^* is injective.

□

Recall: For any space X and abelian group G, $H^k(X;G) \twoheadrightarrow \mathrm{Hom}(H_k(\mathbb{Z}),G)$ (this is an isomorpism if $G = \mathbb{F}$ is a field and X is suitably nice, i.e. cohomology is isomorphic to the dual of homology here). But for $X = M$ an oriented manifold, we have a map

$$H^k(M;\mathbb{F}) \longrightarrow \mathrm{Hom}(H^{n-k}_{\mathrm{ct}}(M,\mathbb{F}),\mathbb{F})$$

coming from the non-degeneracy of the pairing (from Poincaré duality). So for manifolds, the cohomology over a field is instead isomorphic to the dual of compactly supported cohomology of a different degree.

We might then wonder if the same holds for homology. We find that this even holds over $\mathbb{Z}$ (as well as fields):

Theorem 8.3. *Let M be an oriented manifold (of finite type). Then there is a distinguished isomorphism:*

$$\mathscr{D} : H^k_{\mathrm{ct}}(M) \to H_{n-k}(M)$$

(all with coefficients $\underline{\text{in } \mathbb{Z}}$), i.e. this map allows us to go between compact cohomology and homology.

Remark: The result over $\mathbb{Z}$ implies the analogous result over a field $\mathbb{F}$.

The definition of this $\mathscr{D}$ map involves the **cap product**:

Definition 8.3. *For any space X, the* **cap product** $\frown: C_k(X) \otimes C^l(X) \to C_{k-l}(X)$ *is defined by:*

$$([v_0,\dots,v_k],\psi) \longmapsto \psi([v_0,\dots,v_l])[v_l,\dots,v_k]$$

(and is vanishes identically if $l > k$).

As always when defining a map on the chain level we need to know how it interacts with the boundary map to see if it descends to a map of (co)homology. The next lemma deals with the properties of the cap product.

Lemma 8.1. *Let X be any space. Then the cap product $\frown$ satisfies:*

(i) (Relation with boundary map). *For $\varphi \in C^l(X)$ and $\sigma \in C_k(X)$ we have*

$$d(\sigma \frown \varphi) = (-1)^l (d\sigma \frown \varphi - \sigma \frown d^*\varphi).$$

Therefore $\frown$ descends to a map $\frown: H_k(X) \otimes H^l(X) \to H_{k-l}(X)$.

(ii) (Naturality). *If $f : X \to Y$ is a map, $\alpha \in H_k(X)$ and $\beta \in H^l(Y)$ then we have:*

$$f_*(\alpha) \frown \beta = f_*(\alpha \frown f^*(\beta)) \quad \in H_{k-l}(Y)$$

(it turns out this is the only correct naturality statement we can make).

(iii) (Relation to cup product). *If $\sigma \in C_{k+l}(X)$, $\varphi \in C^k(X)$, $\psi \in C^l(X)$, then:*

$$\psi(\sigma \frown \varphi) = (\varphi \smile \psi)(\sigma).$$

(iv) There is a relative cap product

$$C_k(X,A) \otimes C^l(X,A) \to C_{k-l}(X).$$

Proof. (i): Just by direct calculations we have

$$d\sigma \frown \varphi = \sum_{i=0}^{l}(-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{l+1}]}\right)\sigma|_{[v_{l+1},\ldots,v_k]} + \sum_{i=l+1}^{k}(-1)^i \varphi\left(\sigma|_{[v_0,\ldots,v_l]}\right)\sigma|_{[v_l,\ldots,\hat{v}_i,\ldots,v_k]}$$

$$\sigma \frown d^*\varphi = \sum_{i=0}^{l+1}(-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{l+1}]}\right)\sigma\left([v_{l+1},\ldots,v_k]\right)$$

and

$$d(\sigma \frown \varphi) = \sum_{i=l}^{k}(-1)^{i-l} \varphi\left(\sigma|_{[v_0,\ldots,v_l]}\right)\cdot\sigma|_{[v_l,\ldots,\hat{v}_i,\ldots,v_k]}.$$

The result then follows by examining these three expressions.

(ii) and (iii): Just direct checks that they hold at the chain level.

(iv): Suppose we consider $\frown: C_k(A) \times C^l(X,A) \to C_{k-l}(A)$, which we can check vanishes directly (just from the defining expressions). Hence this shows $\frown$ descends to a map $C_k(X,A) \times C^l(X,A) \to C_{k-l}(A)$.

□

Remark: As an idea of where we are heading, suppose M is a closed oriented n—manifold. We have shown then that $H^n(M) = H^n_{\text{ct}}(M) \cong \mathbb{Z}$. In fact, assuming $H_n(M) \cong \mathbb{Z}$, there would then be a corresponding generator, $[M] \in H_n(M)$, called the **fundamental class** of M (think of a collection of simplicies covering M "once"). Then $(\bullet) \frown [M]$, capping with $[M]$, gives a map $H^k(M) \to H_{n-k}(M)$, and this will be the duality isomorphism we are after in Poincaré duality.

However such a $[M] \in H_n(M)$ doesn't exist in the non-compact setting - so we need to work relatively.

Proposition 8.2. *Let M be an oriented n—manifold with orientation generators $\omega_x \in H_n(M, M\backslash\{x\})$ for all $x \in M$.*

Then for every compact set $K \subset M$, $\exists!$ class $\omega_K \in H_n(M, M\backslash K)$ such that the inclusion $(M, M\backslash K) \hookrightarrow (M, M\backslash\{x\})$ sends $\omega_K \mapsto \omega_x$ for all $x \in K$.

[Also, $H_i(M, M\backslash K) = 0$ for $i > n$.]

Proof. Momentarily. ☐

Given this proposition, suppose M is oriented and $K \subset L \subset M$ are compact subsets. Then:

$$\begin{tikzcd}
H_i(M, M\backslash L) \times H^k(M, M\backslash L) \arrow[r, "\frown"] \arrow[d, hook] & H_{i-k}(M) \arrow[d, "\cong"] \\
H_i(M, M\backslash K) \times H^k(M, M\backslash K) \arrow[r, "\frown"] \arrow[u, hook] & H_{i-k}(M)
\end{tikzcd}$$

with the two vertical maps being induced by the inclusions, i.e. i_*, i^* respectively. The uniqueness of ω_K for compact sets K says that $i_*(\omega_L) = \omega_K$ (taking $i = n$ in the above). Then:

$$\omega_K \frown \varphi = i_*(\omega_L) \frown \varphi = \omega_L \frown i^*(\varphi)$$

by naturality of the cap product.

So the map $\varphi \mapsto \omega_K \frown \varphi$ is compatible with the maps defining the direct system of pairs $(M, M\backslash K)$. This means that $\exists$ an induced map

$$\mathscr{D} : H^k_{\text{ct}}(M) \xrightarrow{\frown \omega_\bullet} H_{n-k}(M)$$

i.e. we cap an element of $H^k_{\text{ct}}(M)$ with ω, with the $\bullet$ representing that this is independent of the choice of K by compatibility.

Remark: If M is compact, we can just take $\omega_M = [M] \in H_n(M, M\backslash M) = H_n(M)$, and so $\mathscr{D}$ is exactly $\frown [M]$.

Proof of Proposition 8.2. First note that if ω_A and ω_B exist and are unique, then so does $\omega_{A\cup B}$ by the MV sequence:

$$\underbrace{H_{n+1}(M, M\backslash A \cup B)}_{=0 \text{ by dimensions}} \longrightarrow H_n(M, M\backslash A \cup B) \longrightarrow \underset{\ni \omega_A}{H_n(M, M\backslash A)} \oplus \underset{\ni \omega_B}{H_n(M, M\backslash B)} \longrightarrow H_n(M, M\backslash A \cap B)$$

Uniqueness of ω_A and ω_B says that they both restrict to $\omega_{A\cap B}$, and hence $\exists$ a class $\omega_{A\cap B}$ mapping to (ω_A, ω_B); that class is then unique by exactness.

Now if M is any manifold and $K \subset M$ is compact, we can find a finite number of $K_i \subset D^n \subset M$ compact subsets (in D^n) with D^n a n−dimensional disc such that $K = \bigcup_{\text{finite}} K_i$. So it suffices to prove the proposition when $M = D^n \cong \mathbb{R}^n$.

Note that if $A \subset \mathbb{R}^n$ is compact and convex then by homotopy invariance we have $H_*(\mathbb{R}^n, \mathbb{R}^n\backslash A) \cong H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$. Then the result is straightforward by taking ω_A to be the orientation generator of $H_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$. Therefore as we saw at the start of the proof that we can take unions, we get that the result holds for unions of compact convex sets.

Finally for the general case, let $K \subset \mathbb{R}^n$ be compact. Then we know $\exists R > 0$ such that $K \subset \overline{B_R(0)}$. Moreover since $\overline{B_R(0)}$ is compact and convex, $\omega_{\overline{B_R(0)}}$ is defined by the above. So set:

$$\omega_K := \omega_{\overline{B_R(0)}}|_K$$

(defined via the map $(\mathbb{R}^n, \mathbb{R}^n \backslash \overline{B_R(0)}) \to (\mathbb{R}^n, \mathbb{R}^n \backslash K)$). Then this does restrict to ω_x for all $x \in K$.

So it now suffices to check that no other class has this property, i.e. if $\lambda \in H_n(\mathbb{R}^n, \mathbb{R}^n \backslash K)$ and $\lambda|_x = 0 \in H^n(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$ for all $x \in K$, then in fact $\lambda = 0$. Then this would show that ω_K is well-defined and so we would be done.

So let λ also denote a representing chain for this λ. Then we know $d\lambda$ is a finite union of simplicies in $\mathbb{R}^n \backslash K$. Then one can find a finite union of balls $\{B_j\}_j$ such that

$$K \subset \tilde{K} = \bigcup_j B_j \qquad \text{and} \qquad d\lambda \cap \tilde{K} = \emptyset.$$

So $\lambda \in H_n(\mathbb{R}^n, \mathbb{R}^n \backslash K)$ actually comes from a class in $H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \tilde{K})$ and we already proved the result for unions of convex sets above. So since $\lambda|_x = 0$ for all $x \in K$, we get $\lambda|_x = 0$ for all $x \in \tilde{K}$ (assuming each B_j meets K), and so $\lambda = 0$ follows from uniqueness of $\omega_{\tilde{K}}$.

□

Proof of Poincaré Duality/Theorem 8.2. We prove this by inducting on the number of sets in a finite type good cover of M.

If $M = U \cup V$, then Mayer-Vietoris (on both homology and cohomology and using the $\mathscr{D}$ maps to pass between) gives:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_{\text{ct}}(U \cap V) \arrow[r] \arrow[d, "\mathscr{D}"] & H^k_{\text{ct}}(U) \oplus H^k_{\text{ct}}(V) \arrow[r] \arrow[d, "\mathscr{D}"] & H^k_{\text{ct}}(M) \arrow[r] \arrow[d, "\mathscr{D}"] & H^{k-1}_{\text{ct}}(U \cap V) \arrow[r] \arrow[d, "\mathscr{D}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Then by induction on type, the result will follow from the case of $\mathbb{R}^n$ if the squares in the above diagram commute.

For 2 out of 3 squares this is clear by naturality of the cap product under maps of spaces. So we just need to prove that

$$\begin{tikzcd}
H^k_{\text{ct}}(M) \arrow[r] \arrow[d, "\mathscr{D}"] & H^{k+1}_{\text{ct}}(U \cap V) \arrow[d, "\mathscr{D}"] \\
H_{n-k}(M) \arrow[r] & H_{n-k-1}(U \cap V)
\end{tikzcd}$$

commutes up to a sign which depends only on k.

Exercise: Show that the 5-Lemma holds in this situation (note that if the sign above depended upon the particular element of $H^k_{\text{ct}}(M)$ then this would not be true).

So to prove the signed commutativity, let $M = U \cup V$ and let $K \subset U$ be compact and $L \subset V$ be compact. Then we have a square:

$$\begin{tikzcd}
H^n(M, M\backslash K \cup L) \arrow[r, "d^*_{\text{MV}}"] \arrow[d, "\frown \omega_{K\cup L}"] & \begin{array}{c} H^{k+1}(M, M\backslash K \cap L) \\ \scriptstyle \cong H^{k+1}(U\cap V, U\cap V\backslash K\cap L) \text{ by excision} \end{array} \arrow[d, "\frown \omega_{K\cap L}"] \\
H_{n-k}(M)(d_{MV}) \arrow[r] & H_{n-k-1}(U \cap V)
\end{tikzcd}$$

Write $M = (U\backslash L) \cup (U \cap V) \cup (V\backslash K)$ and represent $\omega_{K\cup L}$ by a sum of simplicies

$$\omega_{K\cup L} = \alpha_{U\backslash L} + \alpha_{U\cap V} + \alpha_{V\backslash K}$$

with respect to this decomposition of M. Then $\alpha_{U\backslash L}$ and $\alpha_{V\backslash K}$ lie in $M\backslash K \cap L$ and so vanish in $C^*(M, M\backslash K \cap L)$, and thus we see that $\alpha_{U\cap V}$ represents $\omega_{K\cap L}$.

Similarly since $\alpha_{V\backslash K}$ vanishes in $C_*(M, M\backslash K)$ we see $\alpha_{U\backslash L} + \alpha_{U\cap V}$ represents ω_K.

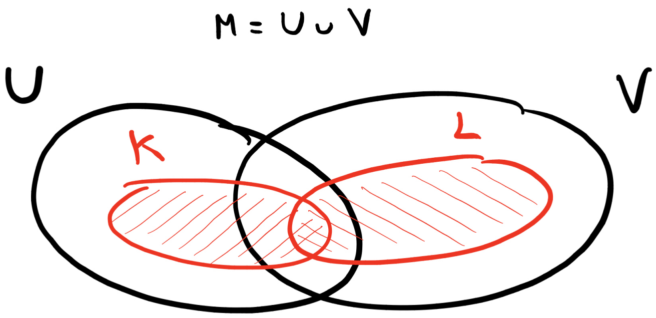


FIGURE 27. An illustration of the decomposition of M.

Now let $\varphi \in H^k(M, M\backslash K \cup L)$. Write

$$\varphi = \varphi_{M\backslash K} - \varphi_{M\backslash L} \quad \in \quad C^*(M, M\backslash K) + C^*(M, M\backslash L).$$

Then by definition we have $d^*_{\text{MV}}(\varphi) := d^*\varphi_{M\backslash K}$. We want to compare

$$\begin{aligned}
&\text{(i)} \;\; \varphi \longmapsto d^*\varphi_{M\backslash K} \longmapsto \alpha_{U\cap V} \frown d^*\varphi_{M\backslash K} \\
&\text{(ii)} \;\; \varphi \longmapsto \underbrace{\alpha_{U\backslash L} \frown \varphi}_{\text{in } U} + \underbrace{(\alpha_{U\cap V} + \alpha_{V\backslash K}) \frown \varphi}_{\text{in } V} \longmapsto d(\alpha_{U\backslash L} \frown \varphi)
\end{aligned}$$

since these represent both ways of moving around the diagram (with (i) being going along the top first whilst (ii) is going down first).

So note:

$$\begin{aligned}
d(\alpha_{U\backslash L} \frown \varphi) &= (-1)^k (d\alpha_{U\backslash L} \frown \varphi - \alpha_{U\backslash L} \frown \underbrace{d^*\varphi}_{=0 \text{ since } \varphi \text{ is a cocycle}}) \\
&= (-1)^k d\alpha_{U\backslash L} \frown \varphi_{M\backslash K}
\end{aligned}$$

because φ vanishes on chains in $M\backslash L$. Then note that since $\alpha_{U\backslash L} + \alpha_{U\cap V} = \omega_K$ we see that $d\alpha_{U\backslash L} + d\alpha_{U\cap V} \in C_*(M\backslash K)$, and then since $\varphi_{M\backslash K}|_{C_*(M\backslash K)} \equiv 0$, we get $\varphi_{M\backslash K} \frown \left(\alpha_{U\backslash L} + d\alpha_{U\cap V}\right) = 0$. So hence we see from the above,

$$d(\alpha_{U\setminus L} \frown \varphi) = (-1)^{k+1} d\alpha_{U\cap V} \frown \varphi_{M\setminus K}.$$

Now compare (from naturality of the cap product):

$$H_* \ni 0 = d(\alpha_{U\cap V} \frown \varphi_{M\setminus K}) = (-1)^k \left(d\alpha_{U\cap V} \frown \varphi_{M\setminus K} - \alpha_{U\cap V} \frown d\varphi_{M\setminus K} \right)$$

i.e.

$$(-1)^{k+1} d\alpha_{U\cap V} \frown \varphi_{M\setminus K} = (-1)^k \alpha_{U\cap V} \frown d^* \varphi_{M\setminus K}$$

which is exactly what we wanted to prove from (i) ad (ii) (up to a sign).

□

Exercise: Using the relation of the cap product and cup product, show Poincaré duality ⇒ "Pairing version" of Poincaré duality. Or (essentially equivalently.), via an analogous induction to the above over a cover, relating the Mayer-Vietoris sequences

$$\begin{tikzcd}
H^k_{\text{ct}}(U\cap V) \arrow[r] \arrow[d] & H^k_{\text{ct}}(U) \oplus H^k_{\text{ct}}(V) \arrow[r] \arrow[d] & \cdots \\
H^{n-k}(U\cap V)^* \arrow[r] & H^k(U)^* \oplus H^k(V)^* \arrow[r] & \cdots
\end{tikzcd}$$

with $\mathbb{F}$—coefficients.

9. Cohomology of Submanifolds

For the rest of the course we return to studing cohomology of submanifolds. In particular, return to chomology classes of submanifolds. There is an obvious notion of a smooth vector bundle over a smooth manifold, $E \to M$, and also of a smooth section s of E.

Lemma 9.1. *Let M be a closed smooth manifold, and $E \to M$ an oriented smooth vector bundle.*

Let $s : M \to E$ be a smooth section such that $s(M)$ is transverse to the zero-section $M \subset E$ (identifying the 0-section by M). Then $Z = s^{-1}(0)$ is a smooth submanifold of E (or of M), it is canonically co-oriented, and

$$\varepsilon_Z = e_E \in H^{rank(E)}(M).$$

Proof. Note firstly that

$$\begin{aligned}\text{codim}_E(Z) &= \text{codim}_E(s(M)) + \text{codim}_E(\text{zero-section}) \quad \text{(as the intersection is transversal)}\\ &= 2\text{rank}(E)\end{aligned}$$

$\Longrightarrow \text{codim}_M(Z) = \text{rank}(E)$, and so the dimensions are consistent.

Since s and the zero-section 0_E intersect transversely, for all $x \in Z$ we have

$$T_xE = T_x(s(M)) + T_x(0_E).$$

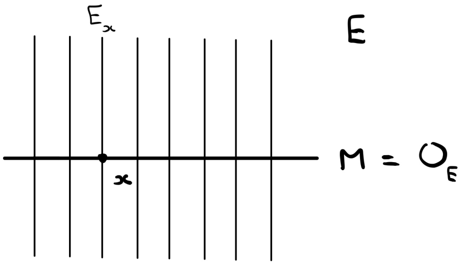


Figure 28. An illustration of the vector bundle E with zero section 0_E.

But also we have $T_xE = T_x0_E \oplus E_x$ and $T_xM = T_xZ \oplus (\nu_{Z/M})_x$. So if $x \in Z$, then $Ds_x : T_xM \to T_xE = T_xM \oplus E_x$ (once again identifying $0_E = M$) is defined by $\xi \mapsto (\xi, \sigma_x(\xi))$ for σ_x linear. Thus we get

$$\sigma_x : (\nu_{Z/M})_x \xrightarrow{\cong} E_x$$

i.e. $s^*E|_z \cong \nu_{Z/M}$, with the isomorphism induced by Ds, the derivative of the section.

Now E orientated $\Rightarrow \nu_{Z/M}$ is oriented ($\Longleftrightarrow$ Z is co-oriented), and

$$\varepsilon_Z = u_{\nu_{Z/M}} = u_{s^*E} = s^*u_E = (\text{inclusion}_{0_E})^*u_E = e_E$$

where we have used naturality of the Thom class, the fact that $s \cong \text{inclusion}(0_E \hookrightarrow E)$, and the definition of the Euler class.

□

Remark: This strengthens our earlier observation that if any (topological) vector bundle $E \to X$ has a nowhere-zero section s, then $e_E = 0$ (i.e. this is the case when the transverse intersection is always $\emptyset$).

Note: Suppose M is a closed smooth manifold, oriented, and $Y \subset M$ is a closed oriented (and hence co-oriented) smooth submanifold. Say $\dim(Y) = k$ and $\dim(M) = n$. Then we now have two ways of associating a cohomology class to Y:

(i) The Thom class $\varepsilon_Y \in H^{n-k}(M)$ defined via $u_{\nu_{Y/M}}$ relative to the tubular neighbourhood identification of $\nu_{Y/M} \cong u_Y \underset{\text{open}}{\subset} M$.
(ii) As Y is a closed oriented manifold, there is a fundamental class $[Y] \in H_k(Y) \cong \mathbb{Z}$, so $(\text{inclusion}_Y)_*[Y] \in H_k(M)$, so
$$\mathscr{D}^{-1}(\text{inclusion}_*[Y]) \in H^{n-k}(M).$$

In the following proposition we see that in fact these two cohomology classes are the same when working over a field.

Proposition 9.1. *Let M be a closed oriented n—manifold and let $Y \subset M$ be a closed oriented k—dimensional submanifold. Assume that Y, M and the inclusion $i : Y \hookrightarrow M$ are smooth. Then:*
$$\varepsilon_Y = \mathscr{D}^{-1}(i_*[Y]) \in H^{n-k}(M, \mathbb{F})$$
where we are working over a $\underline{\text{field}}$ $\mathbb{F}$.

Proof. Let $\hat{\varepsilon}_Y = D^{-1}(i_*[Y])$. Then
$$D(\hat{\varepsilon}_Y) = i_*[Y] \quad\Longrightarrow\quad [M] \frown \hat{\varepsilon}_Y = [Y].$$
Hence over $\mathbb{F}$ we have for all $\alpha \in H^k(M)$,
$$\alpha([M] \frown \hat{\varepsilon}_Y) = \alpha(i_*[Y]) = (i_*\alpha)[Y]$$
i.e. using a slightly different notation,
$$\int_M \alpha \cdot \hat{\varepsilon}_Y = \int_Y \alpha|_Y$$
where for any oriented manifold X of finite type, we write $\int_X$ for the distinguished map $H^n_{\text{ct}}(X) \xrightarrow{\cong} \mathbb{Z}$ (or $\mathbb{F}$).

Moreover the non-degeneracy of the cup product over $\mathbb{F}$ says that $\hat{\varepsilon}_Y$ is <u>characterised</u> by the values:
$$\int_M \alpha \cdot \hat{\varepsilon}_Y = \int_Y \alpha|_Y$$
for $\alpha \in H^k(M)$ (think of $\hat{\varepsilon}_Y$ as like a “Dirac Delta” along Y).

Compare this with: ε_Y is represented by a cocycle c with the properties, for $Y \subset U_Y \underset{\text{open}}{\subset} M$ (with U_Y a tubular neighbourhood of Y):

(a) $c|_{M\setminus U_Y} = 0$

(b) In U_Y, if $U_{Y,y}$ is the image of the fibre $(\nu_{Y/M})_y$ of the normal bundle (so $U_{Y,y} \cong \mathbb{R}^{n-k}$) then

$$c|_{U_{Y,y}} \in H^{n-k}_{\text{ct}}(U_{Y,y})$$

is the (co)orientation generator.

Now

$$\int_M \alpha \cdot \varepsilon_Y = \int_{U_Y} \varphi^* \alpha \cdot u_{\nu_{Y/M}}$$

by definition of the Thom class, where $\varphi : U_Y \hookrightarrow M$, since $\exists$ a cocycle c for ε_Y vanishing on $M \backslash U_Y$.

Claim: $\forall \beta \in H^k(Y)$, we have

$$\int_{U_Y} \pi^* \beta \cdot u_{\nu_{Y/M}} = \int_Y \beta$$

where $\pi : U_Y \cong \nu_{Y/M} \to Y$.

Proof of Claim. Fix a disc $D^K \underset{j}{\subset} Y$, where $j : D^k \hookrightarrow Y$. Then $j_* : H^k_{\text{ct}}(D^k) \to H^k_{\text{ct}}(Y) = H^k(Y)$ (by compactness of Y) is onto. So it suffices to take $\beta \in \text{Im}(j_*)$. But on $\text{Im}(j)$, $(\nu_{Y/M})|_{D^k}$ is the trivial bundle, and so $(\nu_{Y/M})|_{D^k} = D^k \times D^{n-k}$, and $u_{\nu_{Y/M}}$ is just the orientation generator $\varepsilon_{n-k} \in H^{n-k}_{\text{ct}}(\mathbb{R}^{n-k})$.

So our identity reduces to showing that $\varepsilon_k \times \varepsilon_{n-k} \in H^n_{\text{ct}}(\mathbb{R}^n)$ is the generator, under $\mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$, i.e. that

$$H^k_{\text{ct}}(\mathbb{R}^k) \oplus H^{n-k}_{\text{ct}}(\mathbb{R}^{n-k}) \xrightarrow{\times} H^n_{\text{ct}}(\mathbb{R}^n)$$

the cross product, is an isomorphism. However we know that this is true and so this proves the claim.

☐

With this claim we have completed the proof. ☐

Remark: All we want for the rest of the course is that ε_Y (or equivalently $\hat{\varepsilon}_Y$) satisfies

$$\int_M \alpha \cdot \varepsilon_Y = \int_Y \alpha|_Y$$

and that $\varepsilon_{Y \cap Z} = \varepsilon_Y \cdot \varepsilon_Z$ when Y, Z meet transversely. If we define ε_Y via Poincaré duality (which is the usual route) then the integral identity is easy but the relation for transverse intersections is harder to show (whereas defining it the other way as we did makes the integral identity the harder one to establish).

9.1. **Diagonal Submanifolds.**

Let M be a manifold. Write $\Delta \subset M \times M$ for the **diagonal**,

$$\Delta := \{(m,m) \in M \times M : m \in M\}.$$

If M is a smooth manifold, then Δ is a smooth manifold, and we have $\nu_{\Delta/M\times M} \cong TM$, and so an orientation on M co-orients Δ.

Recall that if M is closed (so of finite type) then by the Künneth formula,

$$H^*(M \times M, \mathbb{F}) \cong H^*(M, \mathbb{F}) \otimes H^*(M, \mathbb{F}).$$

Now the cup product is a non-generate pairing on $H^*(M, \mathbb{F})$ (by Poincaré), and so we can pick dual bases

$$\{a_i \in H^{d_i}(M, \mathbb{F})\}_i \quad \text{and} \quad \{b_i \in H^{n-d_j}(M, \mathbb{F})\}_j$$

such that

$$\int_M a_i \cdot b_j = \delta_{ij}.$$

Proposition 9.2. *If $n = \dim(M)$ then we have:*

$$\varepsilon_\Delta = \sum_i (-1)^{d_i} a_i \times b_i \qquad \in H^n(M \times M, \mathbb{F}).$$

Proof. By the non-degeneracy of the cup product, it suffices to prove that for $\xi \in H^p(M; \mathbb{F})$ and $\eta \in H^{n-p}(M; \mathbb{F})$ we have:

$$\langle (\xi \otimes \eta) \cdot \varepsilon_\Delta, [M \times M] \rangle = \left\langle (\xi \otimes \eta) \cdot \left(\sum_i (-1)^{d_i} a_i \times b_i \right), [M \times M] \right\rangle.$$

Now looking at each side:

$$\text{LHS} = \int_{M \times M} (\xi \otimes \eta) \cdot \varepsilon_\Delta = \int_\Delta (\xi \otimes \eta)|_\Delta = \int_M \xi \cdot \eta.$$

$$(\dagger) \qquad \text{RHS} = \int_{M \times M} (\xi \otimes \eta) \cdot \left(\sum_i (-1)^{d_i} a_i \times b_i \right).$$

Now if $|\alpha| = n = |\beta|$, $\alpha, \beta \in H^n(M, \mathbb{F})$, we have

$$\int_{M \times M} \pi_1^* \alpha \cdot \pi_2^* \beta = \left(\int_M \alpha \right) \cdot \left(\int_M \beta \right)$$

using the fact that $[M \times M] = [M] \otimes [M]$ under the Künneth theorem. Using this,

$$(\dagger) = \sum_i (-1)^{d_i} (-1)^{d_i(n-p)} \int_M \xi \cdot a_i \int_M \eta \cdot b_i.$$

This is non-zero only if $p = n - d_i$, i.e. $d_i = n - p$. So we need to show:

$$\int_M \xi \cdot \eta = \begin{cases} \sum_i \underbrace{(-1)^{d_i + d_i^2}}_{=+1} \int_M \xi \cdot a_i \int_M \eta \cdot b_i & \text{if } d_i = n - p \\ 0 & \text{otherwise.} \end{cases}$$

where this sign is $+1$ since $d_i + d_i^2 = d_i(d_i + 1)$ is always even. To see this, since the $\{a_j\}_j$ form a basis of $H^*(M, \mathbb{F})$, let $\eta = a_j$. Then this becomes

$$\int_M \xi \cdot a_j = \left(\int_M \xi_i \cdot a_i \right) \delta_{ji}$$

which is true and what we want. So done.

□

Corollary 9.1 (Gauss-Bonnet Theorem). *Let M be a closed oriented smooth manifold. Then*

$$\int_M e_{TM} = \chi(M).$$

In particular, if $\chi(M) \neq 0$ then every vector field on M is zero somewhere.

Proof. Recall that $\nu_{\Delta/M\times M} \cong TM$, $\underline{\text{so}}$ $e_{TM} = e_{\nu_{\Delta/M\times M}} = \varepsilon_\Delta|_\Delta$ (compare this with $\varepsilon_Y|_Y = e_{\nu_Y}$ always holds).

So,

$$\int_M e_{TM} = \int_\Delta \varepsilon_\Delta|_\Delta = \sum_i (-1)^{d_i} \int_M a_i \cdot b_i$$

and then for $\{a_i\}_i$ and $\{b_i\}_i$ dual bases of $H^*(M;\mathbb{F})$ this becomes

$$= \sum_k (-1)^k \text{rank}(H^k(M,\mathbb{F})) = \chi(M)$$

since as just pick up factors of $(-1)^{d_i}$ for each basis element of $H^k(M,\mathbb{F})$, for all k.

□

Now suppose $Y,Z \subset M$ are closed oriented submanifolds of a closed oriented M (all smooth), then if $\dim(Y)+\dim(Z) = n \equiv \dim(M)$, and if Y,Z intersect transversely, then $Y \cap Z = \{\text{finite set}\}$.

Now define for $x \in Y \cap Z$ the **sign of x** by:

$$\text{sign}(x) := \begin{cases} +1 & \text{if } T_xM = (\nu_{Y/M})_x \oplus (\nu_{Z/M})_x \text{ preserves orientation} \\ -1 & \text{if this decomposition reverses orientation.} \end{cases}$$

Lemma 9.2. *With everything as above we have*

$$\int_M \varepsilon_Y \cdot \varepsilon_Z = \sum_{x\in Y\cap Z} \text{sign}(x).$$

Proof. The LHS is:

$$\int_M \varepsilon_{Y\cap Z} = \int_{Y\cap Z} \varepsilon_{Y\cap Z}|_{Y\cap Z} = \sum_{x\in Y\cap Z} \text{sign}(x)\varepsilon_x = \sum_{x\in Y\cap Z}\text{sign}(x)$$

since $\varepsilon_x = +1$ for all x by definition of local orientation generators. □

Theorem 9.1 (Lefschetz Fixed Point Theorem). *Let M be a closed smooth manifold which is $\mathbb{F}$-oriented. Let $f : M \to M$ be a smooth map with non-degenerate fixed points*$^{\text{(xii)}}$. *Then:*

$$\sum_{x\in \text{Fix}(f)} \text{sign}(x) = \sum_k (-1)^k \text{Trace}\left(f^* : H^k(M,\mathbb{F}) \to H^k(M,\mathbb{F})\right).$$

The quantity on the RHS is often called the ***Lefschetz number of f*** *and denoted $L(f)$.*

Proof. Note that the fixed points of f satisfy: $\text{Fix}(f) = \Delta \cap \Gamma_f$ (or at least the projection of these (x,x) onto M). So we have:

$$\begin{aligned}\sum_{x\in\text{Fix}(f)} \text{sign}(x) &= \sum_{x\in\Delta\cap\Gamma_f} \text{sign}(x)\\ &= \int_{M\times M} \varepsilon_\Delta \cdot \varepsilon_{\Gamma_f} \qquad \text{by Lemma 9.2}\\ &= \int_{\Gamma_f} \varepsilon_\Delta|_{\Gamma_f}\\ &= \int_M (\text{id}\times f)^*\varepsilon_\Delta\end{aligned}$$

where $\text{id}\times f : M \to M\times M$ sends $x \mapsto (x, f(x))$. But note that:

$$(\text{id}\times f)^*\varepsilon_\Delta = \sum_i (-1)^{d_i} a_i \times f^* b_i.$$

Now, $\int_M a_i \cdot f^*(b_i)$ is the (i,i) matrix entry for $f^* : H^k(M,\mathbb{F}) \to H^k(M,\mathbb{F})$ with respect to the basis $\{b_j\}_j$, since if $f^*b_i = \sum_j m_{ij}b_j$ then

$$m_{ii} = \int_M a_i \cdot f^*(b_i)$$

as $\int_M a_i \cdot b_j = \delta_{ij}$. So combining we are done.

□

Remark: If $f : M \to M$ is continuous and $L(f) \neq 0$, then f must have a fixed point by Theorem 9.1 (we extend this to the continuous case since any continuous map can be C^0-approximated (i.e. continuously approximated) by smooth maps, and the property "$\text{Fix}(f) = \emptyset$" is preserved by sufficiently close C^0-approximation, which this gives).

Example 9.1. *Let $f : \mathbb{C}P^{2k} \to \mathbb{C}P^{2k}$ be any map. Then f has a fixed point. In particular, no non-trivial finite group acts freely on $\mathbb{C}P^{2k}$.*

Proof. It suffices to prove any smooth map has a fixed point by the above remark, and thus by the Lefschetz fixed point theorem it suffices to show that $L(f) \neq 0$.

So let $\alpha \in H^2(\mathbb{C}P^{2k},\mathbb{Z}) \cong \mathbb{Z}$. Then $f^*(\alpha) = l\alpha$ for some $l \in \mathbb{Z}$, and so:

$$f^*(\alpha^j) = (f^*(\alpha))^j = l^j\alpha^j.$$

(xii)By this we mean the graph of f, Γ_f and the diagonal Δ (which is the graph of the identity) intersect transversely in $M \times M$.

So we see by definition,

$$L(f) = 1 + l + l^2 + \cdots + l^{2k} = \begin{cases} 2k+1 & \text{if } l = 1 \\ \frac{1-l^{2k+1}}{1-l} & \text{if } l \neq 1. \end{cases}$$

So $L(f) \neq 0$, as required. ☐

Example 9.2. *Let Σ be a closed Riemann surface of genus $g(\Sigma) \geq 2$. Let $f : \Sigma \to \Sigma$ be a holomorphic automorphism. Then if $f \neq \mathrm{id}$, then f acts non-trivially on $H^*(\Sigma_g; \mathbb{Z})$.*

Proof. Let $p \in \mathrm{Fix}(f)$. Take a chart near $p \in \Sigma$. Then locally $f : U \to U$ with $U \cong (D, 0)$ a disc (identifying p with the centre 0), and so f has an isolated fixed point at p, unless $f|_U = \mathrm{id}$ (by the identity theorem from complex analysis).

Moreover at p, f has positive local degree. Now if $\nabla \equiv \Gamma_f$ (i.e. $f \neq \mathrm{id}$), then f has some isolated fixed points, and each contributes positively to $\varepsilon_{\Delta \cap \Gamma_f}$, i.e. $L(f) \geq 0$. But if f acts on cohomology, then

$$L(f) = L(\mathrm{id}) = \chi(\Sigma_g) = 2 - 2g < 0$$

since the genus was > 1. ☐

9.2. **Cobordism.**

Now let M be a closed even-dimensional manifold, and say it is oriented. Let $\dim_{\mathbb{R}}(M) = 2n$. Then $H^n(M, \mathbb{R})$ carries a non-degenerate bilinear form from the cup product, and:

- If $n = 2k+1$, then $\dim(M) = 4k+2$, and the cup product is skew and so is a symplectic form (see Example Sheet 3).
- If $n = 2k$, then $\dim(M) = 4k$ and then $H^{2k}(M; \mathbb{R})$ is a vector space with a non-degenerate symmetric bilinear form and hence this has a **signature**(xiii), denoted $\sigma(M)$ or $I(M)$ (also called the **index** of M).

Exercise: Show that:

(i) $I(M) = \chi(M) \pmod 2$
(ii) $I(M \amalg M') = I(M) + I(M')$
(iii) $I(M \times M') = I(M)I(M')$, where $I(X) = 0$ if $\dim(X) \neq 0 \pmod 4$ [proved via the Künneth formula].

We then have:

(xiii) This is just the signature of a symmetric non-degenerate bilinear form.

Theorem 9.2. *Let M^{4k} be closed and oriented. Then if $M = \partial W^{4k+1}$ is the boundary of a compact $(4k+1)$—manifold W with boundary, then $I(M) = 0$.*

Proof. We will give a sketch later. □

Proposition 9.3. *If M^{2n} is closed and $M = \partial W^{2n+1}$, then $\chi(M)$ is even.*

Proof. Given $M = \partial W$, the **collar neighbourhood theorem** says that ∃ a neighbourhood of ∂W in W which is homeomorphic to $\partial W \times [0, \varepsilon)$, and hence we can define a closed manifold by

$$DW := W \bigcup_{\partial W} W$$

which has $\chi(DW)$ since DW is closed and odd-dimensional (Corollary 8.2).

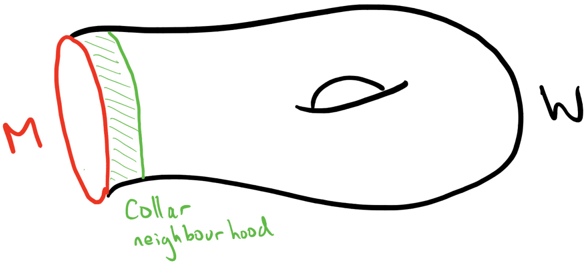


FIGURE 29. An illustration of W, M and the neighbourhood given by the collar neighbourhood theorem.

Now write $DW = U \cap V$, where $U, V \simeq W$ and $U \cap V \simeq \partial W = M$. Then Mayer-Vietoris gives:

$$\cdots \longrightarrow H_{i+1}(DW) \longrightarrow H_i(M) \longrightarrow H_i(W) \oplus H_i(W) \longrightarrow H_i(DW) \longrightarrow H_{i-1}(M) \longrightarrow \cdots$$

This is a chain complex with trivial homology groups, and so the alternating sum of ranks of groups in this sequence vanishes: this says that $\chi(M) = 2\chi(W)$ and so $\chi(M)$ is even.

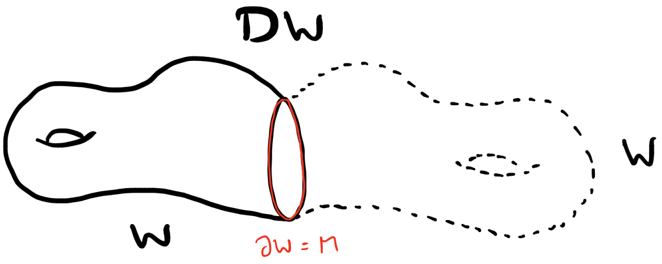


FIGURE 30. An illustration of DW.

□

Corollary 9.2. *$\mathbb{C}P^2$ is not the boundary of a 5-manifold. Similarly $\underbrace{\mathbb{C}P^2 \times \cdots \times \mathbb{C}P^2}_{l \text{ times}}$ is not the boundary of a $(4l+1)$—manifold.*

Proof. Apply Proposition 9.3, since $\chi(\mathbb{C}P^2) = 3$ is odd. ☐

All of this about boundaries of manifolds is useful because of the concept of **cobordism**.

Definition 9.1. *We say that closed smooth n—manifolds M^n and N^n are **cobordant** if $\exists$ a smooth $(n+1)$—manifold W with boundary $\partial W = M \amalg (-N)$, where by $-N$ we mean N with the orientation reversed.*

*Equivalently in the case when M, N are oriented we say that they are **oriented cobordant**.*

Noting that cobordism is an equivalence relation, we define

$$\Omega_n := \frac{\{\text{Orientated smooth } n\text{-manifolds}\}}{\text{Orientated cobordism}}.$$

The operations $(M,N) \mapsto M \amalg N$ and $(M,N) \mapsto M \times N$ descend to Ω_n to make $\Omega_* = \bigoplus_{n\geq 0} \Omega_n$ into a graded ring.

We can then prove that:

$$\Omega_0 = \mathbb{Z}\langle\text{point}\rangle, \quad \Omega_1 = \{0\} \text{ (via a disc)}, \quad \Omega_2 = \{0\} \text{ (via e.g. } \Sigma_2)$$

and

$$\Omega_3 = \{0\}$$

which can be shown using Dehn surgery presentations of 3-manifolds, and

$$\Omega_4 \neq \{0\}$$

since $[\mathbb{C}P^2] \neq 0 \in \Omega_4$, where as usual $[\cdot]$ denotes the cobordism equivalence class. Indeed $I : \Omega_4 \to \mathbb{Z}$, and moreover $I : \Omega_* \to \mathbb{Z}$ is a ring homomorphism.

Now a key result is that whereas classifying manifolds up to diffeomorphism is algorithmically impossible, $\Omega_*^{\mathbb{Q}} \equiv \Omega_* \otimes \mathbb{Q}$ is known, generated as a ring by even-dimensional complex projective spaces (this is related to characteristic classes).

Sketch proof that $I(M^{4k}) = 0$ if $M = \partial W$.

The collar neighbourhood theorem says that every compact set $K \subset (N, \partial N)$, where this pair denotes a manifold N with boundary ∂N, lies in $N \backslash (\partial N \times [0, \varepsilon))$ for some small $\varepsilon > 0$. The construction of classes

$$\omega_K \in H_n(N, N \backslash K)$$

for manifolds generalises to give a relative fundamental class $[N, \partial N] \in H_n(N, \partial N)$, where $\dim(N) = n$. This satisfies:

(a) Under $H_n(N, \partial N) \to H_{n-1}(\partial N)$ we have $[N, \partial N] \mapsto [\partial N]$

(b) $\exists$ a cap product $H^i(N) \to H_{n-i}(N, \partial N)$ defined via: $(-) \frown [N, \partial N]$

(c) These fit into a sequence:

$$\begin{tikzcd}
H^{p-1}(\partial N) \arrow[r] \arrow[d, "\mathscr{D}"', "\cong"] & \overset{\cong H^p_{\mathrm{ct}}(N)}{H^p(N, \partial N)} \arrow[r] \arrow[d, "\mathscr{D}"', "\cong"] & H^p(N) \arrow[r] \arrow[d, "(\star)"] & H^p(\partial N) \arrow[r] \arrow[d, "\mathscr{D}"', "\cong"] & \cdots \\
H_{n-p}(\partial N) \arrow[r] & H_{n-p}(N) \arrow[r] & H_{n-p}(N, \partial N) \arrow[r] & H_{n-p-1}(\partial N) \arrow[r] & \cdots
\end{tikzcd}$$

where $\mathscr{D}$ is the usual Poincaré duality map.

So suppose $M^{4k} = \partial W^{4k+1}$, and let $i : M \hookrightarrow W$ be the inclusion. Then $i^*\left(H^{2k}(W)\right) \subset H^{2k}(M)$. Note:

(i) This is isotopic for the cup product:

$$\langle i^*(\alpha) \cdot i^*(\beta), [\partial W] \rangle = \langle \alpha \cdot \beta, i_*([\partial W]) \rangle.$$

But $[\partial W]$ comes from $[W, \partial W] \in H_{4n}(W, \partial W)$ and so this vanishes since composing two maps in a l.e.s gives 0.

(ii) It's a half-dimensional subspace from:

$$\begin{tikzcd}
H^{2n}(W) \arrow[r, "i^*"] \arrow[d, "\mathscr{D}"] & H^{2n}(M) \arrow[r, "\delta"] \arrow[d, "\mathscr{D}"] & H^{2n+1}(W, M) \arrow[d, "\mathscr{D}"] \\
H_{2n+1}(W, M) \arrow[r, "\partial"] & H_{2n}(M) \arrow[r, "i_*"] & H_{2n}(W)
\end{tikzcd}$$

and so we see $H^{2n}(M) = \mathrm{Im}(i^*) \oplus \mathrm{Im}(\delta) = \mathrm{Im}(i^*) \oplus \mathrm{Im}(i_*)$, and i_*, i^* are adjoint maps.

Then a general fact from linear algebra, which states that if (V, φ) is a vector space with φ a symmetric bilinear form, has a half-dimensional isotropic subspace (like the above), then its signature vanishes!

Since this signature was just the index, we are then done.

□

End of Lecture Course